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Toward a Factually Induced Space-Time Quantum Logic

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In the present work are identified the main features of the algebraic structure with respect to the logical operations, of the set of all the quantum mechanical utterances for which can be specified a factual counterpart and factual rules for truth valuation. This structure is found not to be a lattice. It depends crucially on the spacetime features of the operations by which the observer prepares the studied states and performs measurements on them.

"...atomic propositions, although they cannot contradict, may exclude one another. I will try to explain this. There are functions which give a true proposition only for one value of their argument because—if I may so express myself—there is only room in them for one. Take, for instance, a proposition which asserts the existence of a color R at a certain time T in a certain place P of our visual field. I shall write this proposition "RPT," and abstract for the moment from any consideration of how such a statement is to be further analyzed. "BPT" says that the color B is in the place P at the time T, and it will be clear for most of us here, and to all of us in ordinary life, that "RPT & BPT" is some sort of contradiction (and not merely a false proposition)... How ...does the mutual exclusion RPT and BPT operate? I believe it consists in the fact that RPT as well as BPT are in a certain sense *complete*. That which corresponds in reality to the function "( )PT" leaves room only for one entity—in the same sense, in fact, in which we say that there is room for one person only in a chair. Our symbolism, which allows us to form the sign of the logical product of "RPT" and "BPT," gives here no correct picture of reality.

I have said elsewhere that a proposition "reaches up to reality," and by this I meant that the forms of the entities are contained in the form of the proposition which is about these entities. For the sentence, together with the mode of projection which projects reality into the sentence, determines the logical form of the entities, just as in our simile a picture on plane II, together with its mode of projection, determines the shape of the figure on plane I. This remark, I believe,

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gives us the key for the explanation of the mutual exclusion of RPT and BPT. For if the proposition contains the form of an entity which it is about, then it is possible that two propositions should collide in this very form. The propositions "Brown now sits in this chair" and "Jones now sits in this chair" each, in a sense, try to set their subject term on the chair. But the logical product of these propositions will put them both there at once, and this leads to a collision, a mutual exclusion of these terms. How does this exclusion represent itself in symbolism? We can write the logical product of the two propositions p and q in this way:

p	q	
T	T	T
T	F	F
F	T	F
F	F	F

What happens if these two propositions are RPT and BPT? In this case the top line "TTT" must disappear as it represents an impossible combination. The true possibilities are

RPT	BPT
T	F
F	T
F	F

That is to say, there *is* no logical product of RPT and BPT in the first sense, and herein lies the exclusion as opposed to contradiction. The contradiction, if it existed, would have to be written

RPT	BPT	
T	T	F
T	F	F
F	T	F
F	F	F

but this is nonsense... It is, of course, a deficiency of our notation that it does not prevent the formation of such nonsensical constructions, and a perfect notation will have to exclude such structures by definite rules of syntax. These will have to tell us that in the case of certain kinds of atomic propositions described in terms of definite symbolic features certain combinations of the T's and F's must be left out. Such rules, however, cannot be laid down until we have actually reached the ultimate analysis of the phenomena in question. This, as we all know, has not yet been achieved."

Ludwig Wittgenstein, "Some Remarks on Logical Form" (Aristotelian Society, 1929).

1. INTRODUCTION

1.1. Dedication

I am happy to dedicate this contribution to Professor Henry Margenau, whose views—quite specially those concerning the projection postulate and the problem of joint probabilities—have played an important role in the evolution of my own ideas.

1.2. What Is Researched

It is currently admitted as an immutable evidence that the logical structure of the ensemble of the propositions from quantum mechanics "is"—in some absolute way—that of a nonmodular lattice isomorphic to the lattice of the closed subspaces of the Hilbert space of a system. In this work we indicate the main lines of an alternative view. We bring into evidence that a logical structure is relative to the requirements chosen (explicitly or implicitly). The particular structure obtained here is entailed by a systematic reference to the factual definability of the considered utterances and of their truth values and it is not isomorphic to the lattice of the closed subspaces of the Hilbert space of a system.

Consider an utterance u that concerns some designatum. In general the relation between the utterance and the designatum is itself not easily definable. *A fortiori*, the truth value of the utterance is in general not easily determined. Now, quantum mechanics is a physical theory. This incites us to proceed as follows. An utterance u occurring inside quantum mechanics will be called a "factual proposition," or, in short a proposition, if and only if:

> There exists a physical counterpart of u specifiable independently of the utterance u itself: for any u we require systematically a global statement of the type "u, and u points toward the factual counterpart $F(u)$." (1)

> There exists an operational, *instructional* specification of some method for *comparing* the factual counterpart $F(u)$ with the utterance u, and for *deciding* whether the result of the comparison has to be expressed by the truth value "true" or by the truth value "false." We shall then speak of "*factual* truth valuation" and of truth values "factually true" (1_F, factual confirmation) and "factually false" (0_F, factual falsification). (2)

The conditions (1)–(2) are called "factuality conditions" for a proposition. The factuality conditions lead outside the quantum mechanical formalism. They treat this formalism as just a pool of utterances—concepts, assertions—expressed in a language that makes use of certain mathematical notions, but the formal structure where these mathematical notions are inserted is here *completely ignored*. Among the utterances only the assertions are considered, *exclusively* for examining the satisfaction of the conditions (1)–(2). If these cannot be met, we shall say that a factual truth value cannot be defined. The considered utterance will then be regarded as irrelevant and will not be taken into account. This entails that there is *no point* in using a logical connective which, starting from one or more propositions in the sense (1)–(2), yields a new utterance that violates the condition (1) or (2) or both: in that case this new utterance, accordingly to (1)–(2), *would have to be dropped*. So a given logical connective will be introduced only *relatively* to *those* propositions in the sense (1)–(2) from which it generates propositions in the sense (1)–(2). We shall speak of "factually definable logical connectives" or "factuality-preserving logical connectives."

The approach outlined in this work has also other specificites:

(a) The probabilistic organization of the quantum theory is regarded as basic with respect to the logical features of the theory. It is utilized as a datum and a reference (Ref. 1 and Appendix). It yields an insight into the *physical content* of the different types of quantum mechanical propositions, and this content interacts with the conditions of factuality (1), (2). (Therefore it would be very efficient to start with the Appendix).

(b) We distinguish radically between [factual truth value of a *probabilistic* proposition] and [probability of the factual truth value of a proposition (probabilistic or not)]. A probabilistic proposition that asserts a probability *less* than 1 can be found to be "logically" certain or true (factual truth value 1_F, no observed exception to the asserted subunitary probability value), and a probabilistic proposition asserting a probability 1 ("probabilistic certainty") can be found to be "logically" false (factual truth value 0_F). As a consequence we are free to *include* the probabilistic specifications into propositions thus considering "probabilistic propositions," and *then* to research *yes-no* truth valuations for these probabilistic propositions. This entails a radical scission with the previously developed approaches.

(c) We distinguish explicitly between the three different levels of conceptualization that are involved: individual, statistical, and probabilistic.

The specificities a, b, c form together a framework where the way in which logical qualifications can be coherently connected with probabilistic qualifications becomes crystal clear.

Our aim is to investigate the logico-algebraic structure of the set of all the quantum mechanical factual propositions, that is, the algebraic structure induced into this set by the factuality-preserving logical connectives. The "factual logic" of quantum mechanics—in as much as this locution indicates the calculus of tautological transformations of the formally true quantum mechanical factual propositions—is not yet researched here.

It might be found strange and difficult to have to eliminate from one's mind any previously achieved formal structure (Hilbert-space, orthomodular lattices, etc.). We apologize and ask the favor of attention: we try to convey an attempt that might throw some new light on the preceding ones.

2. ON THE LOGICO-ALGEBRAIC STRUCTURE OF THE SET OF THE FACTUAL PROPOSITIONS FROM QUANTUM MECHANICS

We shall try to scan the quantum theory and to identify *exhaustively* all the types of factual propositions from it. We put apart the well-known definitions of formal descriptors from the quantum theory (state vectors, observables, etc.) as well as the so called "postulates" or "axioms" which associate a specified physical meaning to these formal descriptors: These are neither veritable physical postulates nor propositions in our sense, they just introduce the language of quantum mechanics. Throughout the scanning the probabilistic organization of the quantum theory, as exposed in the Appendix, is used as a reference-background.

We begin by the quantum mechanical probabilistic propositions. These express all the quantum mechanical probabilistic predictions.

2.1. Probabilistic Postulating Propositions

2.1.1. Atomic Probabilistic Postulating Propositions. The most specified probabilistic assertions from the quantum theory can be formulated as follows:

AΠP: *The probability density* $\pi(\psi, \omega_j)$ *that the eigenvalue* ω_j *be obtained if a measurement evolution for the observable* Ω *is performed on a state with state vector* $|\psi\rangle$, *is* $\pi(\psi, \omega_j) = |\langle u_j | \psi\rangle|^2$.

Such an assertion will be qualified as "atomic (A), probabilistic (Π), postulating (P)": AΠP, which emphasizes that the considered proposition concerns only *one* definite state vector $|\psi\rangle$, *one* observable Ω, and one eigenvalue ω_j of Ω (atomic) and that it is posited by the quantum theory to be *certainly* true (postulating). (The notation $|\langle u_j | \psi\rangle|^2$ is well known.)

The Factuality Conditions. Is an utterance of the type AΠP a factual proposition in the sense (1)–(2)?

Content. *What* does an utterance of the type AΠP *assert*? Neither the studied state $|\psi\rangle$, nor the measurement evolution for the considered observable Ω, nor the eigenvalue ω_j. It only *involves* these three specifications. What is asserted is exclusively the quantum mechanical probability assignation $\pi(\psi, \omega_j) = |\langle u_j|\psi\rangle|^2$, *if* the "conditions" $|\psi\rangle$, Ω, ω_j are realized. This assertion involves a stratified conceptual attitude:

— On a first level of conceptualization it is—implicitly—presupposed that *a priori* one could conceive of *any* value for the probability (density) $\pi(\psi, \omega_j)$, *not necessarily the quantum mechanical one*, $|\langle u_j|\psi\rangle|^2$.

— On a *meta* level of conceptualization it is asserted—implicitly and reflexively—that the *particular* probability density of which the value is that one—$\pi(\psi, \omega_j) = |\langle u_j|\psi\rangle|^2$—that is calculated inside the formalism of quantum mechanics, will emerge, *certainly*: An explicit statement about the value of a *probability* coalesces with an implicit assertion of *certitude* about the truth value 1_F for that very statement; so a *reflexive* certitude entailed by (equivalent to) the status of a postulate assigned in quantum mechanics to any utterance of the type AΠP.

We shall uphold that *it is this implicit assertion of the truth value* 1_F *that is offered by the theory for truth valuation.*

But for the moment, let us first examine whether the factuality conditions (1)–(2) are fulfilled.

Factual Counterpart F(u). The factual definition of a quantum mechanical probability density $\pi(\psi, \omega_j)$ can be researched only inside an ensemble of a very big number N' of a very big number N of reiterations of a corresponding "elementary quantum mechanical chain experiment" (eqmce)[1] (pp. 1390–95 and Appendix). An elementery quantum mechanical chain experiment consists of a sequence [operation P_ψ of preparation of a state with state vector $|\psi\rangle$; measurement evolution M_Ω for the observable Ω; registration of a "needle position" V_k corresponding to some eigenvalue $\omega_k = f(V_k)$]. We label such a sequence by the symbol $P_\psi - M_\Omega - V_k$, $k \in K$, K an index set. When an eqmce is reiterated a big number N of times, *various* results V_k, $k \in K$, are obtained, which in general are *distinct* from the particular result V_j tied with the eigenvalue ω_j envisaged in the *studied* AΠP utterance u. The factual probability density $\pi(P_\psi, M_\Omega, V_j) = \pi(V_j)$ for this particular result *can* nevertheless be constructed, as follows. Inside the ensemble of *all* the N reiterations of the individual chain experiment $P_\psi - M_\Omega - V_k$, the number $n(V_j)$ of the realizations of the particular "fiber" $P_\psi - M_\Omega - V_j$ is counted. Once obtained, the number $n(V_j)$ is then

referred to the total number N of experiments, for estimation of the relative frequency $[n(V_j)/N]$. This *whole* procedure is *itself* reiterated a big number of times N', with increasing values N, to "test"[2] (pp. 283–285) the existence as well as the value of a limit for the ratio $[n(V_j)/N]$ when N "tends toward infinity." *This limit is defined as the "factual probability"* $\pi(P_\psi, M_\Omega, V_j)$. The whole procedure constitutes the factual counterpart F(u) of the considered AΠP utterance u, which thus is defined. So the condition (1) of factuality is fulfilled.

Factual Truth Valuation. By factual examination of F(u) it is determined whether the numerical equality $\pi(P_\psi, M_\Omega, V_j) = |\langle u_j|\psi_1\rangle|^2$ obtains satisfactorily. If (with an accepted predefined approximation) the answer is yes, the factual truth *meta* value $1_F{}^{(2)}$ has been obtained. If (with respect to that same accepted predefined approximation) the answer is no, the factual truth *meta* value $0_F{}^{(2)}$ has been obtained. A truth *meta* value in *any* case: Indeed in the first case, when $\pi(P_\psi, M_\Omega, V_j) = |\langle u_j|\psi_1\rangle|^2$, what shall we conclude? Shall we only say that the numerical value of the particular probability density $\pi(P_\psi, M_\Omega, V_j)$ factually determined in that investigation, involving the particular conditions symbolized by $P_\psi - M_\Omega - V_k$, has been found to be equal to the particular corresponding number $|\langle u_j|\psi\rangle|^2$ calculated inside quantum mechanics? Obviously not: On the *basis* of this particular finding we shall say something *else* and much more important, namely that the quantum mechanical *postulate* (or law) $\pi(P_\psi, M_\Omega, V_j) = |\langle u_j|\psi_1\rangle|^2$—for any $|\psi\rangle$, Ω, ω_j—has "not been falsified" (K. Popper). While if $\pi(P_\psi, M_\Omega, V_j) \neq |\langle u_j|\psi_1\rangle|^2$, what shall we say? Only that, in the particular investigation that has been performed, an inequality has been found between the two independently constructed numbers $\pi(P_\psi, M_\Omega, V_j)$ and $\langle u_j|\psi_1\rangle|^2$? Obviously we shall again conclude something *else.* We shall say that the *status* of a postulate assigned inside quantum mechanics to *any* AΠP utterance has been factually falsified by this particular experiment. Which leads us *outside* quantum mechanics. So *we conclude systematically by reference to the implicit and reflexive assertion of the (first-order) truth value* 1_F *contained in the examined* AΠ*P utterance u:* The factual truth value obtained by us is regarded as a factual *meta* truth value of second order qualifying the truth value 1_F implicitly asserted in the examined AΠP utterance u. As announced, what is offered by an AΠP proposition for truth valuation is the implicit assertion of the truth value 1_F for the explicit assertion of the quantum mechanical probabilistic prediction $\pi(\psi, \omega_j) = |\langle u_j|\psi\rangle|^2$. This is only a particular manifestation of a quite general and well known fact of fundamental importance brought forth by the logical study of language (Russell, Tarski, etc.), namely that there must exist a hierarchy of languages and that the words "true" and "false", applied to sentences from any given language, are themselves words belonging to a language of higher order. Which entails as a consequence the existence of a language of lower order where the words "true" and "false" do not occur.

The factuality conditions (1)–(2) being fulfilled, *any AΠP utterance is a proposition in our sense.*

Mutual Compatibility. In what follows there appears a first specific consequence of the factuality conditions (1), (2): *any* two AΠP propositions are *compatible.*

We have remarked that the probability $\pi(P_\psi, M_\Omega, V_j)$ can be factually defined only inside a very big number N' of reiterations of measurements of the relative frequencies $[n(V_k)/N]$ (with increasing numbers N) corresponding to the elementary quantum mechanical chain experiments $P_\psi - M_\Omega - V_k$, $\forall k \in K$, K an index set. During the process by which this factual definition is built in order to confront it with the quantum mechanical prediction, it is not possible to filter out in advance, to program the emergence of exclusively the particular fibers $P_\psi - M_\Omega - V_j$ that involve the particular outcome V_j tied with the eigenvalue $\omega_j = f(V_j)$ from the studied AΠP proposition: an eqmce has an unpredictable outcome V_k. The bunches of identical fibers $P_\psi - M_\Omega - V_j$ can only *a posteriori* be selected inside the N' realized ensembles of (increasingly many) N fibers $P_\psi - M_\Omega - V_k$, each one of these ensembles containing (in general) also all the other possible fibers with eigenvalues $V_k \neq V_j$. But such an *a posteriori* selection can equally be done for *any* index $k \neq j$, "at the same time," i.e., *using the same N' pools of increasingly many N fibers.* In this sense, the factual counterpart of every AΠP proposition is organically *tied* (by a common genesis; by reference to the same N', N; by the norm condition $\Sigma_k |\langle u_k | \psi \rangle|^2 = 1$) to the factual counterpart of all the other AΠP propositions corresponding to the same pair (P_ψ, M_Ω):

> On the descriptional level where are located the AΠP propositions (a *meta* metalevel of description with respect to the level of description where are placed the individual fibers $P_\psi - M_\Omega - V_k$[21]) (pp. 283) operates a "*meta*-metatime" with "*meta* metasimultaneities" and "*meta*-metasuccessivities." And *with respect to this meta-metatime*—the only one significant in this context—there is *no mutual exclusion* between the factual counterparts of two different bunches of fibers $P_\psi - M_\Omega - V_j$ and $P_\psi - M_\Omega - V_k$ corresponding both to the same pair (P_ψ, M_Ω). There is, quite the contrary, an *unavoidable* "meta-metacoexistence" between them, thus factual *compatibility.*

A fortiori there is no mutual exclusion either between the factual realizations of two AΠP propositions corresponding to two distinct fibers $P_\psi - M_\Omega - V_k$ bringing in either two distinct preparations P_ψ and $P'_\psi \neq P_\psi$, or two distinct measurement evolutions M_Ω and $M'_\Omega \neq M_\Omega$, or both. Since there is no restriction on the number of replicas of the studied system S to be used, nothing prevents us from working simultaneously with a number of replicas as huge as we desire and to perform on one subset experiments $P_\psi - M_\Omega - V_k$ while on another one we perform experiments $P'_\psi - M'_\Omega - V'_k$. So we introduce the symbol of factual compatibility $\leftrightarrow_F$ and we write

$$\forall a, b \in \{A\Pi P\}, \qquad a \leftrightarrow_F b$$

2.1.2. The Boolean Lattice $L_B(\Pi P)$ of the Probabilistic Postulating Propositions. We drop now the condition of atomicity (only one fiber $P_\psi - M_\Omega - V_k$) to consider any quantum mechanical probabilistic postulating (ΠP) proposition.

Logical Conjunction. In consequence of the mutual compatibility $a \leftrightarrow_F b, \forall a, b \in \{A\Pi P\}$, the operation of logical conjunction (product, intersection) of any two or more AΠP propositions *does* possess a factual counterpart, in the most usual sense: It points toward the (meta-meta) *simultaneous* factual realizability of the conditions (defined above) that permit one to determine all the various factual probability densities $\pi(P_\psi, M_\Omega, V_j)$, to be compared with the corresponding quantum mechanical densities $|\langle u_j | \psi \rangle|^2$, which entails that the separate factual truth values of any two or more AΠP propositions can be all determined—as specified above—(meta-meta)*simultaneously.* These separate factual truth values entail then for the conjunction itself, accordingly to the usual rule, the factual truth value true if all the "intersected" AΠP propositions are factually true, and the factual truth value false if this is not the case. Thus,

> We *do* define an operation of factual logical conjunction Λ_F between any number of quantum mechanical AΠP propositions.

Logical Sum, etc.. Since we dispose of a concept of (factual) logical product, the set $\{\Pi P\}$ of all the quantum mechanical probabilistic postulating propositions can be further organized in the usual way: definition of a partial order and of a logical sum, introduction of an absurd proposition $\varnothing$ and of a trivial one I, definition of a greatest lower bound and a smallest upper bound for any subset of ΠP propositions, and an orthocomplement of any ΠP proposition.

Structure. So we finally obtain a *boolean* lattice.

> *The logical operations endowed with factual counterpart that can be defined for the atomic probabilistic postulating propositions generate on the set $\{\Pi P\}$ of all the probabilistic postulating propositions an algebra that is a boolean lattice $L_B(\Pi P)$.*

This result is far from being trivial. It is of course obvious that on the subset of the probabilistic postulating propositions from *one* branch of a *given* probability tree all the logical connectives are factuality-preserving and so form a boolean logical algebra. But *a priori* it is not obvious at all that the same holds concerning the probabilistic postulating propositions from different branches of a tree, or from different trees. This does happen only *relatively* to the factuality conditions (1), (2) imposed here and because we recognized that quantum mechanics, when it associates a probabilistic qualification with a proposition, *includes* that qualification *in* the proposition and furthermore posits—*implicitly*—for the proposition thus obtained the logical truth value true.

To understand clearly how this works, compare with the "formally induced" approaches initiated by von Neumann[3] (p. 249). According to von Neumann *any* quantum mechanical proposition is of only one same general form, namely: "The result of a measurement of the quantum mechanical observable Ω operated on a system with state vector $|\psi\rangle$ is an eigenvalue ω_j of Ω belonging to this or that closed interval $\Delta\omega$ of the spectrum of Ω, $\omega_j \in \Delta\omega$." Now, such a form does not *include* a probabilistic qualification. In von Neumann's approach—on *any* quantum mechanical proposition—a probabilistic qualification is superimposed from outside the proposition, by asserting *and* accepting without any further truth-control that the considered proposition is "true with probability $|P_{\Delta\omega}|\psi\rangle|^2$" ($P_{\Delta\omega}$: the projector onto the closed subspace $\Delta\omega$ of the Hilbert space of the system). Thereby logical, yes-no truth qualifications, and probability qualifications, *coalesce* forming a logicoprobabilistic qualification that does not distinguish between [truth value of a *probabilistic* proposition] and [probability of the truth value of a proposition] (probabilistic or not); this coalesced concept furthermore *fuses* with the quantum mechanical *formalism* (via the calculus with projectors $P_{\Delta\omega}$) which *imprisons a priori inside the formalism the researched logical structure* (and entails well known problems concerning the connection between logical products and products of projectors).

On the contrary, applying the factuality conditions (1), (2) to quantum mechanical propositions that include a probabilistic qualification and—globally and implicitly (Russell)—are posited to be true, we are led to a clear distinction between the *probabilistic* qualification contained in such a proposition, and its factual *logical* (yes-no) truth valuation; we remain free of any subjection to the formalism of quantum mechanics, free to organize just a *face-à-face* between the propositions of the quantum theory, and facts; and the factual logical truth-valuation of a probabilistic qualification brings us on a meta-metalevel of investigation where there is no restriction concerning the number of the involved replicas of the studied

system, and *this* is what entails absense of incompatibility between *any* two probabilistic propositions.

2.1.3. Equivalences Inside the Boolean Lattice $L_B(\Pi P)$. The probabilistic qualification involved in any quantum mechanical postulating proposition entails the possibility of a hierarchy of three different sorts of equivalences:

— concerning propositions from one same branch of a probability tree (norm condition)

— concerning propositions from two different branches of one same tree (transformation theory, relating the probability measures from different branches of the same tree (Appendix))

— concerning propositions from different trees [the principle of superposition, relating different trees and probability measures that, inside these different trees, correspond to the same observable (Appendix)].

Equivalence Involving One Branch. Consider a proposition $a \in \{\Pi P\}$ that asserts the probability $\pi[\psi, q(a)]$ concerning an event (in the sense of probabilities) $q(a)$ that belongs entirely to the algebra of events τ_F from *one* branch of *one* tree (see Appendix). Let $q'^{\tau}(a)$ be the *ensemblistic* complement of the event $q(a)$ with respect to that algebra of events τ_F. The normalization imposed by definition upon any probability measure together with the definition of a "sum of events" and of the probability of such a sum entail then also the factual truth of the proposition $(a)'^{\tau}$ that asserts for the event $q'^{\tau}((a)$ the probability $\pi[\psi, q'^{\tau}(a)] = 1 - \pi[\psi, q(a)]$. So we can write the following "logico-probabilistic equivalence":

$$a \Leftrightarrow_F (a)'^{\tau}$$

($\Leftrightarrow_F$: factually equivalent)

Equivalence Involving Different Branches of One Tree. We have shown[1] (pp. 1401–1401 and Appendix) that the common initial genesis of two distinct branches belonging to the same probability tree entails a probabilistic meta-dependence between the probability measures from these two branches, considered *globally*. How does this translate in the language of logic?

In the probability tree corresponding to the operation of state preparation P_ψ, consider the branch which involves an observable Ω_1. Let $a[\pi(\psi, \Omega_1)]$ be the probabilistic postulating proposition that asserts the *whole* probability measure $\pi(\psi, \Omega_1)$ from this branch. The proposition $a[\pi(\psi, \Omega_1)]$ can be represented as the factual logical intersection of all the *atomic* propositions $a[\pi(\psi, \omega_{j1})$ corresponding to a given eigenvalue ω_{j1} of

$\Omega_1 : a[\pi(\psi, \Omega_1)] \Leftrightarrow_F \Lambda_{F, j \in J} a[\pi(\psi, \omega_{j1})]$. Furthermore let $b[\pi(\psi, \omega_{k2})]$ be the atomic probabilistic postulating proposition which asserts the elementary probability $\pi(\psi, \omega_{k2})$ for the eigenvalue ω_{k2} of an observable Ω_2 that does not commute with the observable Ω_1. According to the quantum mechanical transformation theory we have $\pi(\psi, \omega_{k2}) = |\Sigma_j \alpha_{kj} c(\psi, \omega_{j1})|^2$ (where $|c(\psi, \omega_{j1})| = \sqrt{\pi(\psi, \omega_{j1})}$ and the α_{kj} are the transformation coefficients). This, with the defined notations, translates in the following "logico-probabilistic equivalence":

$$\forall \omega_{k2}), (b[\pi(\psi, \omega_{k2})] \Leftrightarrow_F (\Lambda_{F, j \in J} a[\pi(\psi, \omega_{j1})] \Leftrightarrow_F a[\pi(\psi, \Omega_1)]$$

Equivalence Involving Different Trees. Consider the superposition state vector $|\psi_{ab}\rangle = \lambda_a |\psi_a\rangle + \lambda_b |\psi_b\rangle$ that involves three trees founded upon three distinct operations of state preparation $P_{\psi a}$, $P_{\psi b}$ and $P_{\psi ab} = G(\lambda_1, \lambda_2, P_{\psi a}, P_{\psi_b})$ (Appendix). Let $(ab)[\pi(\psi_{ab}, \omega_j)]$, $a[\pi(\psi_a, \omega_j)]$ and $b[\pi(\psi_b, \omega_j)]$ be the AΠP propositions asserting, respectively, the elementary probability densities $\pi(\psi_{ab}, \omega_j)$, $\pi(\psi_a, \omega_j)$ and $\pi(\psi_b, \omega_j)$ for the eigenvalue ω_j of the observable Ω in, respectively, the states $|\psi_{ab}\rangle$, $|\psi_a\rangle$ and $|\psi_b\rangle$. According to quantum mechanics we have $\pi(\psi_{ab}, \omega_j) = |\lambda_a \langle u_j | \psi_{1a}\rangle + \lambda_b \langle u_j | \psi_{1b}\rangle|^2$. This, with the defined notations, translates in the following "logico-probabilistic equivalence":

$$(ab)[\pi(\psi_{ab}, \omega_j)] \Leftrightarrow_F a[\pi(\psi_a, \omega_j)] \Lambda_F b[\pi(\psi_b, \omega_j)]$$

2.2. Individual Actualized-Potential Propositions

We quit now the probabilistic level of conceptualization and go down to the *individual* level: the level where each proposition concerns just *one* definite replica S_i (*i*: individualized) of the studied system S. Throughout what follows, only this one and the *same* individualized replica S_i is involved, only *its* Hilbert space.

2.2.1. Atomic Individual Actualized-Potential Propositions. The generation on S_i of some definite fiber P_ψ-$M_\Omega - V_k$ is a *lasting* process. We can consider this process from various temporal points of view. We can *insert* our attention—and also our action—*inside* the process, in two different ways: immediately after the accomplishment of only the monadic fragment P_ψ, or after the accomplishment of the diadic fragment $P_\psi - M_\Omega$. At the limits, we can examine the whole fiber either before its beginning or after its accomplishment: When the actualized fragment is of zero length the still potential fragment is a whole fiber; when the actualized fragment is a whole fiber, the potential fragment is of zero length. To each one of these temporal viewpoints there corresponds a type of atomic propositions expressing an association between the already actualized fragment, with

one (any one) among all the as yet nonrealized fragments that are still available as potentialities. A proposition of such a kind will be called an "individual atomic actualized-potential" proposition, and the generic symbol for it will be AI(|) (A: atomic, i.e., concerning one definite sequence of specifications; I: individual, tied to the one individualized replica S_i of the studied system; the first space inside the parenthesis: receptacle for the specification of the actualized fragment; the bar: sign of "present," of temporal separation between past and future; the second space: receptacle for the specification of the envisaged potential fragment).

Imagine first an actualized fragment of 0 length. We are then in the presence of an ensemble of possible hypotheses concerning still entirely potential fibers $P_\psi - M_\Omega - V_k$. Each one of these hypotheses admits of a formulation of the following type:

> $AI(0 | P_\psi - M_\Omega - V_j)$: *The operation of state preparation P_ψ will be realized on the replica S_i of the studied system S, and the initial state $|\psi\rangle$ produced by it will be subject to a measurement evolution for the observable Ω that will yield the registration of the eigenvalue $\omega_j = f_\Omega(V_j)$.*

Because in this limiting case the still potential fragment $P_\psi - M_\Omega - V_j$ is triadic, the ensemble of all the $AI(0 | P_\psi - M_\Omega - V_j)$ propositions is a "triply branching" ensemble. To begin with, any operation of state preparation P_ψ is possible, which, starting from 0, yields a first branching. Then each already actualized fragment P_ψ leaves open the possibility to choose to perform upon S_i a measurement evolution for any quantum mechanical observable Ω, which generates a second branching. Finally each choice of a measurement evolution for some fixed observable Ω will still leave open the possibility of future realization of any one among the eigenvalues ω_j of that observable, which generates a third branching.

Consider now a monadic actualized fragment P_ψ. It introduces an ensemble of AI(|) propositions each element of which admits of a formulation of the type:

> $AI(P_\psi | M_\Omega - V_j)$: *The operation of state preparation P_ψ has been realized on the replica S_i of the studied system S, and the initial state $|\psi\rangle$ produced by it will be followed first by a measurement evolution for the observable Ω and then by the registration of the eigenvalue $\omega_j = f_\Omega(V_j)$.*

So this time we are in the presence of a definite hypothesis concerning the future evolution of the process started on S_i by a definite operation of state preparation P_ψ. Because the still potential fragment $M_\Omega - V_j$ is now diadic, the ensemble of all the $AI(P_\psi | M_\Omega - V_j)$ propositions is an only doubly branching ensemble.

In an anologuous way, a dyadic fragment $P_\psi - M_\Omega$ actualized on S_i introduces the simply branching ensemble of AI(|) propositions of the type:

$AI(P_\psi - M_\Omega | V_k)$: *The operation of state preparation* P_ψ *and a measurement evolution for the observable* Ω *have been realized on the replica* S_i *of the studied system S, and this will be followed by the registration of the eigenvalue* $\omega_j = f_\Omega(V_j)$.

Finally, if the potential fragment becomes of zero length we are in the presence of an entirely actualized fiber $P_\psi - M_\Omega - V_k$ produced by an already entirely achieved elementary quantum mechanical chain experiment. The corresponding AI(|) proposition is of the limiting type opposed to the type $AI(0 | P_\psi - M_\Omega - V_j)$:

$AI(P_\psi - M_\Omega - V_j | 0)$: *The operation of state preparation* P_ψ *has been realized on the replica* S_i *of the studied system S, and the initial state* $|\psi\rangle$ *produced by it has been followed first by a measurement evolution for the observable* Ω *and then by the registration of the eigenvalue* $\omega_j = f_\Omega(V_j)$.

There is no more branching, the whole process is unique, entirely frozen in the past.

At a first sight it might seem that the definition of the concept of AI(|) proposition (or of any equivalent of it) is unnecessary, artificial, too analyzed. But in fact this concept is quite essentially involved in our reasonings concerning all the quantum mechanical experiments. As soon as qualifications and representations are attempted concerning compatible or noncompatible observables, or the multiplicity of the possible eigenvalues of an observable, more or less explicitly but irrepressibly the *individual* nature of a quantum mechanical chain-experiment—tied to one given replica S_i of the studied system S—and its *duration* which entails the possibility of actualized-potential specifications, *are* both present in the mind.

Now, this individual nature entails certain mutual exclusions that are tied *only* and *always* with individuality. But if the conceptual situation is not explicated and systematically dealt with, these exclusions are erroneously assigned to other causes. Then, as it appears below, they are ignored in certain cases in which they do arise, and assumed in others where in fact they are absent. It is strongly amputating to jump directly to probabilistically qualified propositions (as in the approaches initiated by von Neumann). The individual substratum of these has to be studied in its own right.

The Factuality Conditions. We have *a priori* called "proposition" an individual "actualized-potential" utterance. But is such an utterance indeed a factual proposition in the sense (1)–(2)?

Content. Again we ask preliminarily: What does an AI(|) utterance assert? In contradistinction to the atomic probabilistic postulating propositions AΠP, an AI(|) utterance no longer implies a reflexive assertion of certain truth. We are now in the presence of an individual experiment concerning which the quantum theory contains no predictive laws, a mere hypothesis concerning the evolving inner structure of an individual fiber. This hypothesis is what is asserted by an AI(|) utterance and what it offers for (factual) truth valuation. So in this case the quantum theory permits *a priori* both confirmation and *falsification.* The physicist is now placed on the first level of logical qualification.

Factual Counterpart F(u) and Factual Truth Valuation. The factual counterpart of an AI(|) utterance *splits* into counterparts *relative* to the three various sorts of qualifications from it, P_ψ, M_Ω, V_j. Correlatively the factual truth valuations split into subvaluations concerning these various qualifications.

Consider first a qualification q_a—a specified P_ψ, or M_Ω, or V_j—asserted in the *actualized* fragment of the studied $AI(q_a|)$-utterance *u*. By inspection of the factual situation $F(u)$ toward which the proposition points, it can be found that q_a has been realized, or that it has not. In the first case we say that $AI(q_a|)$ has been found to be factually true *with respect* to q_a and we write: *u* has factual truth value $1_F/q_a$. In the second case we say that $AI(q_a|)$ has been found to be factually false *with respect* to q_a and we write: *u* has factual truth value $0_F/q_a$ (which, in contradistinction to what happens in the case of an AΠP utterance, does not contradict quantum mechanics).

Consider now a qualification q_p—again a specified P_ψ, or M_Ω, or V_j—asserted in the still *potential* fragment from the studied $AI(|q_p)$ utterance. This potential fragment contains hypotheses on possible *future* outcomes. So the qualification q_p can be tested only by *continuing* to actualize the partially actualized fiber $P_\psi - M_\Omega - V_j$ envisaged in the studied $AI(|q_p)$ utterance until the stage of actualization corresponding to the nature of q_p is reached. (The truth value of a proposition is founded on its ontological content: so long as this ontological content is only potential, this truth value also is only potential). If after actualization the factual result is found to be precisely that one from the envisaged qualification q_p, we say that $AI(|q_p)$ has been found to be factually true *with respect* to q_p and we write: *u* has factual truth value $1_F/q_p$. If not, we say that $AI(|q_p)$ has been found to be factually false *with respect* to q_p and we write: *u* has factual truth value $0_F/q_p$. (Again this does not contradict the quantum theory.)

If the factual truth value 1_F or the factual truth value 0_F is obtained uniformly with respect to all the qualifications involved in the studied AI(|) utterance, we say "absolutely" that, respectively, this utterance is factually true (has factual truth value 1_F) or factually false (has factual truth value 0_F).

Since we have been able to specify a factual counterpart for any AI(|) utterance as well as for its truth valuation, we say that such an utterance is indeed a factual proposition.

Compatibility. We arrive now at the core of the whole present approach.

The already actualized fragment from a *given* AI(|) proposition is *unique*, definitive, and frozen in the unique past of the individualized replica S_i. So any qualification q_a that is different from that one (of the same nature) that is contained in this actualized fragment, is *incompatible* with the considered AI(|) proposition.

As for the still potential fragment, it is envisaged *by definition* as a continuation of the unique already actualized fragment, a continuation *on the already engaged individualized replica S_i* of the studied system S. So, whatever given qualification q_p from the still potential fragment is subsequently actualized, or even if it is only *imagined* to be subsequently actualized, *this* actualization eliminates the possibility of an (effective or only imagined) actualization of any *other, different* qualification q_p of the same nature. Therefore any potential qualification q_p that is different from that—of the same nature—envisaged in the studied AI(|) proposition, is incompatible with *this* proposition. It asserts another future for S_i while the individualized replica S_i disposes of only one future, exactly as it disposes of only one past. So:

Two different AI(|) propositions, because they are tied with one same individualized replica S_i of the studied system S, are factually incompatible (even if only potentially). They cannot both exist simultaneously. PRIOR to any question of truth valuation, the factual EXISTENCES of two different AI(|) propositions are mutually exclusive with respect to the only "one-individual-time" that is available for the one replica S_i that is tied with these two propositions. So we write:

$$a \;\exists\!\!\!\!\not\;_{F}\; b$$

where the symbol $\exists\!\!\!\!\not\;_{F}$ *means; "factual mutual existential exclusion."*

Note that:

— The factual existential mutual exclusion *vanishes* as soon as we consider any two propositions that are *not* individual (not tied with one individualized replica S_i of the studied system S), *even* if these propositions concern two *noncommuting* observables Ω and $\Omega' \neq \Omega$. Indeed, as soon as two or more replicas of S are allowed, we have already *left* the individual level of conceptualization and *moved* onto the *statistical* level. In that case each replica introduces its own space-time receptacle. But this is *not* the case to which applies what is called in quantum mechanics "incompatible experiments." *Two "incompatible experiments" in the sense of quantum mechanics—whatever the temporal viewpoint—are always posited (more or less explicitly) to concern ONE same individualized replica.*

— The factual existential mutual exclusion is not specific of qualifications imposing two noncommuting observables; it holds for *any* difference between two AI(|): different P_ψ or different V_j *as well.*

These remarks show that the source of the factual existential mutual exclusion of two different AI(|) propositions is just the *individual* character, this and nothing else. The surreptitious confusion between factual existential mutual exclusion and on the other hand incompatibility between measurement evolutions corresponding to two noncommuting observables (or the corresponding formal fact of the noncommutativity of these observables), has to be eradicated.

Two different AI(|) propositions, because they are tied with the same individualized replica S_i of the studied system, are like Wittgenstein's Brown and Jones. The function "AI(|)S_i" (read: AI(|) asserted concerning S_i) and Wittgenstein's function "( )PT" behave in the same way. One can paraphrase closely Wittgenstein's way of expressing the situation concerning his own function "( )PT":

> "The function 'AI(|)S_i'" is in a certain sense *complete*, it leaves room only for the realization of one fiber $P_\psi - M_\Omega - V_j$, in the same sense, in fact, in which we say that there is room for one person only in a chair. A symbolism which would allow us to form the sign of the logical product of two distinct AI(|) propositions would not give a correct picture of reality. If a proposition contains the form of an entity which it is about, then it is possible that two propositions should collide in this very form. Two distinct AI(|) propositions, each, in a sense, try to tie their own fiber $P_\psi - M_\Omega - V_j$ to the *one* considered individualized replica S_i of the studied system S. But the logical product of these propositions will tie both these fibers at once to S_i, and this leads to a collision, a mutual exclusion of these terms".

The individual character of an AI(|) proposition plays the same role as Wittgenstein's place-time restriction PT (here-and-now). The oneness of the replica S_i—the individuality condition symbolized by I in the sign AI(|)—entails precisely Wittgenstein's restriction PT.

2.2.2. The Non-Boolean Algebra $\mathfrak{a}_{NB}$I(|) of Individual Actualized-Potential Propositions. We drop now the atomic character for an individual actualized-potential proposition, so also the corresponding symbol A, but the *individual* character is *maintained.* Thereby we bring into consideration the set {I(|)} of all the "individual actualized-potential propositions I(|)." The propositions from this set concern *all* the unique individualized replica S_i of the studied system S already considered in the examination of the AI(|) propositions. What is the content of the set {I(|)}? What algebraic structure generate on the set {I(|)} the logical operations endowed with factual counterpart?

Absence of a Factuality-Preserving Logical Product. We first define explicitly the content on the set {I(|)}. To begin with, consider the set {AI(|)} of all the atomic individual actualized-potential propositions. Let Σ {AI(|)} be the set of all the subsets from {AI(|)}. The elements A, B, C,... of Σ {AI(|)} are (in general) *sets* of atomic propositions, not atomic propositions. The operation of ensemblistic inclusion $\supseteq$ defines a partial order in Σ {AI(|)} and this—if combined with the definition of a logical sum—permits one to associate by the usual procedures a definite individual actualized-potential proposition a, b, c,... to each element A, B, C,..., respectively, from Σ {AI(|)}:

The proposition a that "corresponds" to $A \in \Sigma$ {AI(|)} is by definition

$$a = \bigvee_F u_k, \qquad \forall u_k \in A \tag{3}$$

(i.e., a is factually true as soon as one—any—of the atomic propositions $u_k \in A$ is factually true). And if B contains A we say that "a implies factually b," i.e., that whenever a is factually true, b is factually true. We write

$$[B \supseteq A] \Leftrightarrow_F [a \Rightarrow_F b] \tag{4}$$

where :$\supseteq$: ensemblistic inclusion; $\Rightarrow_F$:factual implication; $\Leftrightarrow_F$: factually equivalent (already defined). Indeed, in consequence of (3) the operation denoted $\Rightarrow_F$ satisfies the usual properties of logical implication

$$a \Rightarrow_F a$$

$$a \Rightarrow_F b \text{ and } b \Rightarrow_F a \text{ implies } a \Leftrightarrow_F b$$

$$a \Rightarrow_F b \text{ and } b \Rightarrow_F c \text{ implies } a \Rightarrow_F c$$

The propositions a, b, c,... constructed accordingly to (3), (4) are the individual actualized-potential propositions from the set {I(|)}. We have by construction

$$\{I(|)\} \supseteq \{AI(|)\} \tag{5}$$

so (3) and (4) inject into {I(|)} satisfaction of the factuality conditions (1), (2) already verified for the atomic propositions from {AI(|)}.

Furthermore nothing prevents one from introducing into the set {I(|)} an absurd proposition $\varnothing$ and a trivial proposition I as well as a complement for each proposition *and* "subcomplements" relativized to the various qualifications from it: When the considered qualification specifies an operation of state preparation, the corresponding subcomplement leads to a probability tree different from that where the initial proposition is located; when the qualification specifies a measurement evolution the corresponding subcomplement leads to another branch of the same tree; when the qualification specifies an eigenvalue the corresponding subcomplement leads to another fiber of the same branch of the same tree (Appendix).

So far, so good. But what about a logical product? In the absence of factual compatibility, a logical product between two different AI(|) propositions would be devoid of factual counterpart, so accordingly to our rules it cannot be defined. So:

> *A factuality-preserving logical product cannot be generally defined in {I(|)}. Our symbol* ⋈̸ *of mutual existential exclusion between any two different AI(|) propositions points explicitly toward this impossibility.*

The introduction of this symbol follows Wittgenstein's remark that it is "a deficiency of our notation that it does not prevent the formation of such nonsensical constructions, and a perfect notation will have to exclude such structures by definite rules of syntax. These will have to tell us that in the case of certain kinds of atomic propositions described in terms of definite symbolic features certain combinations of the T's and F's must be left out. Such rules, however, cannot be laid down until we have actually reached the ultimate analysis of the phenomena in question. This, as we all know, has not yet been achieved."

On Probabilities Versus Logic. The explicit formulation of the AI(|) propositions and of their properties leads now to "the ultimate analysis of the phenomena in question". And there *we touch the common roots of probabilities and logic.* Indeed the situation is related in a very interesting way with the fact that each AI(|) proposition is tied with a quantum mechanical elementary event in the sense of probabilities: Inside a given random phenomenon—by the very definition of this concept—each *elementary* event is subject to a *strictly* individualizing here-and-now condition (so, if a "system" is involved, only one definite replica of this system can come in). And, as is well known, *the theory of probabilities simply does NOT define a—probabilistic—product for two distinct elementary events.* It defines a probabilistic product only at the cost of an *abstraction* made of the strictly individualizing conditions, so at the cost of a *vanishing* of the elementary character: either "here" is dropped, or "now," or the (implicit) specification of one given replica of the studied system (or experiment), which amounts to the *same* and shifts systematically on the level of statistical conceptualization. Indeed, the theory of probabilities offers in two ways a definition of a probabilistic product, by construction of a product space on the universe of elementary events from the studied space, or by examination of the probability measure on the events from the algebra of the space. In the first case the probabilistic product—very surreptitiously—*ceases* in fact to concern the elementary events from the *initial* space: each elementary product-event from the product-space is brought forth by a product-experiment that *duplicates* the initially considered experiment, therefore requiring either to drop at least one of the two here-and-now conditions, or (which amounts to the same) to drop the oneness of the involved replica of the studied system or experiment. In the second case again the definition skips the elementary events: it applies exclusively to those events from the algebra that are "compatible" in the sense of probabilities, i.e., that possess *in common* one or more elementary events (hence involve two or more distinct replicas of the studied system, when a "system" is involved). So two elementary events reconsidered as events in the algebra of the space cannot be compatible in the sense of probabilities, since their ensemblistic intersection is systematically void.

In this way it appears that probabilities and logic possess common operational roots: the strictly individualizing here-and-now conditions play a *foundational* role in the calculus of probabilities *as well as* in the calculus of a factually induced spacetime logic. But when one considers directly and exclusively propositions in the sense of logic, without reference to probabilities, the factual mutual exclusion—prior to any question of truth valuation—between the existences of those atomic propositions that are placed on the level of strictly individual conceptualization tends to remain hidden. Pushed by implicit aims of "generality," one tends to abstract away the space-time constraints, thereby skipping the level of individualized conceptualization. Correlatively one tends to posit the *universal* existence of a factual counterpart for the logical product of *any* two propositions.

On the other hand, in the particular case of quantum mechanics, certain *consequences* of the factual mutual exclusion between the existences of those atomic propositions that are placed on the level of strictly individualized conceptualization, *have* been perceived. They have been perceived because they have led to certain manifestations inside the formalism, namely the peculiar algebra of the set of the closed subspaces of the Hilbert space of the studied system. But the source of these manifestations has been insufficiently identified. It has been posited to consist of what is called

"incompatibility of noncommuting observables": It remained *unclear* that, as soon as the restriction to only one replica is imposed, the *factual* mutual exclusion realizes *equally* for different operations of state preparation or for different eigenvalues of one same observable, while as soon as the restriction to only one replica is dropped, this exclusion vanishes even if noncommuting observables are considered. So the possibility of factual exclusions between the *existences* of individual atomic propositions, "as opposed to contradiction" and hindering the definition of a logical product, was skipped. This generated a tendency toward imposition of logical structures which yield a loose or distorted representation of the underlying *semantic* situation.

But the possibility of mutual factual exclusions between the existences of individual atomic propositions, entailing simply absence of a factual counterpart for a logical product, becomes striking when one considers propositions *each* one of which is tied with an elementary event from some probability space: Though only implicitly, the probabilistic description is much more specific than the logical one concerning the hierarchy of the levels of conceptualization and their connection with space-time constraints[2] (pp. 277–290). In particular—in its own language—the probabilistic description offers *formalized* expressions of the fact that *as soon as we consider two or more different $AI(|)$ propositions, surreptitiously we have already shifted on the statistical metalevel where propositions involving ensembles of replicas of the studied system are placed.* This stronger connection with factuality has been favored by the fact that the probabilistic conceptualization is free of meta-questions concerning truth valuations, so it was less disturbed in the search and expression of the operational roots of the formalization. Here lies one of the advantages of building a logical representation of quantum mechanics by explicitly referring each step to the probabilistic organization of the theory, *outside* any preestablished formalism.

Structure. In this situation we do not see what significant and factuality-preserving definition could be introduced for the greatest lower bound of any family of propositions from $\{I(|)\}$. So:

> *The logical operations endowed with factual counterpart that can be defined for the individual actualized-potential propositions generate on $\{I(|)\}$ an algebra $\alpha_{NB}I(|)$ that is non-boolean* because it does not contain a universally defined logical product *(not because it is non-modular). So, this algebra is* not even a lattice.

2.3. Remarks Concerning the Statistical Level

Inside the quantum theory the statistical utterances are considered only inasmuch as they constitute a necessary substratum for the probabilistic propositions (see Section 2.1). As such they appear in a form that can be obtained from the limiting form of the propositions $AI(P_\psi - M_\Omega - V_j|0)$ where the condition I of *individuality* is dropped, which leads to entirely actualized fibers $(P_\psi - M_\Omega - V_j)$ concerning *any* replica of the studied system S. When also the condition of atomicity is dropped, one obtains, in the usual way, a richer set of statistical utterances (partial order, sum, etc.). It is obvious from the discussion developed in Section 2.1 for the probabilistic propositions that:

— The statistical utterances are propositions in our sense.

— The algebraic structure of the set of the statistical propositions, with respect to the factuality-preserving logical operations, is boolean. It is in fact a substructure *contained* in the boolean lattice $L_B(\Pi P)$ of the probabilistic postulating propositions.

In this way is ensured the *connection* between the set $\{I(|)\}$ of the individual actualized-potential propositions, and the set $\{\Pi P\}$ of the probabilistic postulating propositions.

2.4. The Global Structure

All the quantum mechanical propositions have been now considered. Indeed the quantum mechanical probabilistic postulating propositions contain in them all the statistical propositions, in the sense specified just above. And these, by imposition of the condition of individuality and shiftings of the insertion of the time-origin (the observer's present), generate all the individual actualized-potential propositions. So when the two sets $\{\Pi P\}$ and $\{I(|)\}$ have been examined, no other quantum mechanical factual propositions are left. We can then conclude as follows.

The algebraic structure of the set of all the factual propositions from quantum mechanics, with respect to the factuality-preserving logical operations, is a non-boolean algebra that is not a lattice, defined on the set $\{\Pi P\} \cup \{I(|)\}$.

This non-boolean algebra will have to be worked out so as to preserve all the semantic specificities involved: hierarchical structure, relative negations and complements for the individual propositions, absence of conjunction for distinct atomic individual propositions. It will be very interesting to really understand the connections between this factually induced structure, commanded by the space-time characteristics of the involved entities (processes), and the well-known more "formally induced" approaches of von Neumann,[3] Jauch,[4] Piron,[5] Mittelstaedt,[6] and many others. (In particular, Mittelstaedt's approach, though it comes to different conclusions concerning the structure, has much in common with our approach.)

Anyhow, the logicoalgebraic structure induced on the set $\{\Pi P\} \cup \{I(|)\}$, by the factuality conditions (1), (2) is *not* isomorphic to the algebraic structure of the set of the closed subspaces of the Hilbert space of the studied system. When

— the factuality conditions (1), (2) are required,

— the probabilistic organization of the quantum theory is utilized as a basic datum,

— a radical distinction is made between the three involved levels of conceptualization, individual, statistical, probabilistic,

— a radical distinction is made between [factual truth value of a *probabilistic* proposition] and [probability of the factual truth value of a proposition (probabilistic or not)],

there emerges a logico-algebraic structure that mirrors directly the own organization of the semantic matter toward which points the formalism of the quantum theory. This structure cannot be lodged in the geometrico-algebraic structure of the *formalism* of the theory. The geometrico-algebraic structure of the formalism distorts the own organization of the semantic matter toward which the formalism points.

Die Probe

Zu einem seltsamen Versuch erstand ich mir ein Nadelbuch. Und zu dem Buch ein altes zwar, doch äusserst kühnes Dromedar. Ein Reicher auch daneben stand, zween Säcke Gold in jeder Hand. Der Reiche ging alsdann herfür und klopfte an die Himmelstür. Drauf Petrus sprach: "Geschrieben steht, dass ein Kamel weit eher geht durchs Nadelöhr als du, du Heid, durch diese Türe gross und breit!" Ich, glaubend fest an Gottes Wort, ermunterte das Tier sofort, ihm zeigend hinterm Nadelöhr ein Zuckerhörnchen als Douceur. Und in der Tat! Das Vieh ging durch, obzwar sich quetschend wie ein Lurch! Der Reiche aber sah ganz stier und sagte nichts als: "Wehe mir!"

Christian Morgenstern, *Galgenlieder Der Gingganz*

APPENDIX. SUMMARY OF PRECEDING RESULTS ON THE PROBABILISTIC ORGANIZATION OF QUANTUM MECHANICS

We indicate briefly a previously constructed view[1] concerning the probabilistic organization of quantum mechanics (sometimes with changed or simplified notations). Novelty will be found in the global lines, not in the details.

Formal and Factual Quantum Mechanical Probability Chains

Formal Probability Chains. Consider a pair $(|\psi\rangle, \Omega)$ where $|\psi\rangle$ is the state vector assigned at the time t to the considered microsystem S, and Ω

is a hermitian operator representing a quantum mechanical dynamical observable. For each such pair the quantum theory defines a probability density law $\pi(\psi, \omega_j)$, $j \in J$, (J an index set) for the emergence of an eigenvalue ω_j of the observable Ω when a measurement of Ω is performed on S in the state $|\psi\rangle$. Namely it is postulated that a specified probability density can be calculated by use of the formula $\pi(\psi, \omega_j) = |\langle u_j | \psi \rangle|^2$, $\forall j \in J$, where $|u_j\rangle$ is the eigenvector corresponding to the considered eigenvalue ω_j (for simplicity we suppose a nondegenerate situation). This probability measure is integrated in a formal "probability chain," i.e., a sequence (random phenomenon) ⇝ [a probability space on the universe of elementary events produced by that random phenomenon], that can be symbolized by

$$(\psi, \Omega, \{\omega_j\}) \rightsquigarrow [\{\omega_j\}, \tau, \pi(\psi, \Omega)]$$

($(\psi, \Omega, \{\omega_j\})$: symbol for the random phenomenon that involves the state vector $|\psi\rangle$ and the dynamical observable Ω, and that produces by reiteration the universe $\{\omega_j\}$ of formal elementary events; τ: the total algebra on $\{\omega_j\}$; $\pi(\psi, \Omega)$: the probability density law on τ determined, via the law of total probabilities, by the elementary probability density law $\pi(\psi, \omega_j) = |\langle u_j | \psi \rangle|^2$.

Factual Probability Chains. To each formal probability chain there corresponds a factual probability chain

$$(\mathrm{P}_\psi, \mathrm{M}_\Omega, \{V_j\}) \rightsquigarrow [\{V_j\}, \tau_\mathrm{F}, \pi(\mathrm{P}_\psi, \mathrm{M}_\Omega)]$$

(P_ψ: the operation of state preparation that produces the state with state vector $|\psi\rangle$; M_Ω: an individual measurement evolution for the observable Ω; V_j: "needle position" of a macroscopic device D_Ω for measurements of the observable Ω; $(\mathrm{P}_\psi, \mathrm{M}_\Omega, \{V_j\})$: the random penomenon that involves the operation P_ψ of state preparation and the individual measurement evolutions M_Ω and which by reiteration produces the universe of elementary events $\{V_j\}$; τ_F: the total algebra on $\{V_j\}$ (F: factual); $\pi(\mathrm{P}_\psi, \mathrm{M}_\Omega)$: the probability measure on τ_F.

Connection. Each eigenvalue $\omega_j \in \{\omega_j\}$ from a formal chain is posited to be calculable as a function $\omega_j = \mathrm{f}_\Omega(V_j)$ of that observed "needle position" V_j from the factual chain that is labeled by the same index $j \in J$. Furthermore, each factual elementary probability density $\pi(\mathrm{P}_\psi, \mathrm{M}_\Omega, V_j)$ is posed to be numerically equal to the corresponding formal elementary probability density: $\pi(\mathrm{P}_\psi, \mathrm{M}_\Omega, V_j) = \pi(\psi, \omega_j) = |\langle u_j | \psi \rangle|^2$.

Elementary Quantum Mechanical Chain Experiments

A sequence $\mathrm{P}_\psi - \mathrm{M}_\Omega - V_j$ is called an "elementary quantum mechanical chain-experiment" (eqmce). It possesses a remarkable unobservable

depth wherefrom emerges into the observable only the extremity V_j that contributes to the construction of the factual observable universe of elementary events $\{V_j\}$. Each observable quantum mechanical "event" (*non*elementary) from an algebra τ_F from a factual quantum mechanical probability space contains inside its semantic substratum all the unobservable sequences of operations and processes forming the corresponding elementary quantum mechanical chain experiments that end up with the registration of the needle positions V_j contained in that event. So *any* quantum mechanical prediction concerns either an elementary quantum mechanical chain experiment, or a union of such experiments. *The elementary quantum mechanical chain experiments are the "fibers" out of which is made the factual substance of the quantum theory.*

Quantum Mechanical Probability Trees

We *fix* now an operation of state preparation P_ψ. Consider the ensemble of all the factual probability chains determined by P_ψ *and* the set of *all* the dynamical observables $\Omega_1, \Omega_2, \Omega_3, \ldots$ defined in quantum mechanics. The probability chains from this ensemble constitute together a certain *unity*, in consequence of their common provenance P_ψ. What is the *space-time* structure of this unity?

For all the chains from the considered unity, the space-time support of the operation of state preparation P_ψ is the *same*, but not also for all the individual measurement evolutions M_Ω involved in this unity. The ensemble of these evolutions *splits* into sub-ensembles $\mathrm{M}_\mathrm{X}, \mathrm{M}_\mathrm{Y}, \ldots$ of mutually "compatible" processes corresponding to mutually commuting observables. The textbooks contain very confusing considerations concerning "successive measurements" of compatible observables (versus the projection postulate). But in fact the notion of *successive* measurements simply is irrelevant for compatible observables: Each *one* measurement evolution from one sub-ensemble, say M_X, is such that *each* registration of a value V_j of the "needle position" of the corresponding macroscopic device D_X permits one to calculate—from the *unique* datum V_j—via a set of *various* theoretical connecting definitions $\omega_{j1} = f_1(V_j)$, $\omega_{j2} = f_2(V_j), \ldots$, *all the different* eigenvalues ω_{j1}, $\omega_{j2}\ldots$ labeled by the *same* index j, for, respectively, all the observables $\Omega_1, \Omega_2, \ldots$ measurable by a process belonging to the class M_X. This entails that for all the commuting observables corresponding to *one* same class M_X the physical process leading to the registration of a value of the "needle position" of the device D_X can be one common process covering just one space-time support, while this is *not* possible for two noncommuting observables belonging to two distinct classes M_X and M_Y: *This is what is commonly designated as "Bohr-*

complementarity", *nothing else*. This entails that, globally, the unity constituted by the ensemble of all the factual probability chains corresponding to a fixed operation of state preparation P_ψ possesses *a branching, tree-like space-time structure*.

Let us symbolize this tree-like structure by $\mathscr{T}(\psi)$ and let us call it a "quantum mechanical probability tree" (in short, a probability tree). So the operations of state preparation P_ψ define, on the set of all the quantum mechanical probability chains, a *partition* in probability trees. *A fortiori*, they define such a partition also on the ensemble of all the elementary quantum mechanical chain experiments out of which the factual quantum mechanical probability chains are made.

Figure 1 provides a simplified example of a probability tree, with only four observables $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ and three branches. The individual measurement evolutions M_{12} (abbreviated notation) correspond to two commuting observables Ω_1, Ω_2, while M_3 and M_4 correspond to two non-commuting observables Ω_3, Ω_4. The notations $[\,]_{12}$ and $[\,]_3$, $[\,]_4$ indicate the factual, observational probability spaces corresponding, respectively, to the measurement evolutions M_{12}, M_3, and M_4 realized on the state represented by $|\psi\rangle$. Each one of the probability spaces $[\,]_n$, $n = 12, 3, 4$, emerges—with respect to an origin of times *reset to 0 after each eqmce*—at some corresponding specific time t_{12} (i.e., $t_2 - t_1$), t_3 (i.e., $t_3 - t_1$), and t_4 (i.e., $t_4 - t_1$). The branch corresponding to Ω_1, Ω_2 contains a very big number of fibers $P_\psi - M_\Omega - V_{k12}$ each one of which ends up with *one* needle position $V_{k12} \in \{V_{j12}\}$ that permits one to calculate *two* distinct corresponding eigenvalues, $\omega_{k1} \in \{\omega_{j1}\}$ and $\omega_{k2} \in \{\omega_{j2}\}$, via two different theoretical definitions $\omega_{k1} = f_1(V_{k12})$, $\omega_{k2} = f_2(V_{k12})$. The branch corresponding to Ω_3 contains a big number of fibers $P_\psi - M_\Omega - V_{k3}$ each one

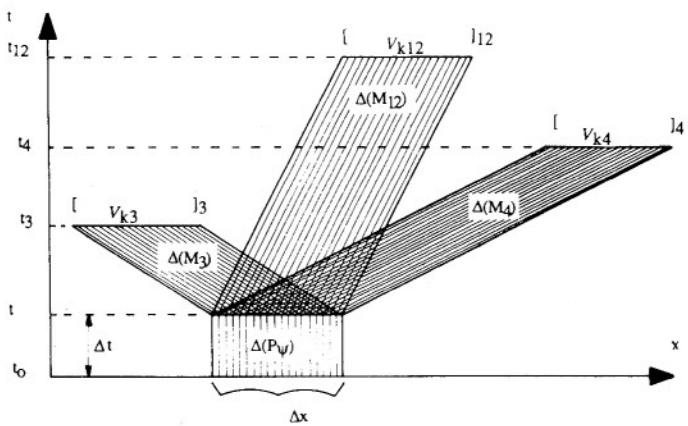

Fig. 1. A quantum mechanical probability tree $\mathscr{T}(\psi)$.

of which ends up with a needle position $V_{k3} \in \{V_{j3}\}$ that permits one to calculate a *unique* corresponding eigenvalue $\omega_{k3} \in \{\omega_{j3}\}$ *via* a theoretical connecting function $\omega_{k3} = f_3(V_{k3})$. Similarily the branch corresponding to Ω_4 contains a big number of fibers $P_\psi - M_\Omega - V_{k4}$ each one of which ends up with a needle position $V_{k4} \in \{V_{j4}\}$ that permits one to calculate a *unique* corresponding eigenvalue $\omega_{k4} \in \{\omega_{j4}\}$ *via* a theoretical definition $\omega_{k4} = f_4(V_{k4})$. So the space $[\,]_{12}$ is endowed with more specifications than the spaces $[\,]_3$ and $[\,]_4$.

In each one fiber of the tree the initial phase, of state preparation, covers the same space-time domain $\Delta(P_\psi) = \Delta x \Delta t$ (the common space-time trunk of the tree), with $\Delta t = t - t_0$. The subsequent phase of measurement evolution covers, for each one fiber, a space-time domain $\Delta(M_{12})$ in the case of an evolution M_{12} corresponding to the two commuting observables Ω_1, Ω_2, or one of the two distinct space-time domains $\Delta(M_3)$, $\Delta(M_4)$ in the case of, respectively, a measurement evolution M_3 corresponding to the observable Ω_3, or a measurement evolution M_4 corresponding to the observable Ω_4.

A quantum mechanical probability tree is a remarkably comprehensive metastructure of probability chains. Most of the fundamental algorithms of the quantum mechanical calculus which combine *one* normed state vector with the dynamical operators representing the quantum mechanical observables can be defined *inside*—any—*one* tree $\mathscr{T}(\psi)$:

— the mean value of an observable Ω in a state with state vector $|\psi\rangle$: $\langle\psi|\,\Omega\,|\psi\rangle$, $\forall\,|\psi\rangle$, $\forall\Omega$

— the uncertainty theorem, for any pair of observables:

$$\langle\psi|\,(\Delta\Omega_1)^2\,|\psi\rangle\langle\psi|\,(\Delta\Omega_2)^2\,|\psi\rangle \geqslant |\langle\psi|\,(i/2)(\Omega_1\Omega_2 - \Omega_2\Omega_1\,|\psi\rangle| = (1/2)(h/2\pi), \qquad \forall\Omega_1, \Omega_2$$

— the principle of spectral decomposition (expansion postulate)

$$|\psi\rangle = \sum_j c(\psi, \omega_j)\,|u_j\rangle, \forall\,|\psi\rangle, \forall \mathrm{A}\colon \mathrm{A}\,|u_j\rangle = \omega_j\,|u_j\rangle,$$

$(c(\psi, \omega_j)$: the expansion coefficients$)$

which permits to calculate the probability density $\pi(\psi, \omega_j)$ *via* the probability postulate $\pi(\psi, \omega_j) = |\langle u_j|\psi\rangle|^2 = |c(\psi, \omega_j)|^2$

— and finally, the whole quantum mechanical "transformation theory" from the basis of an observable Ω_1 to that of an observable Ω_2

$$c(\psi, \omega_{k2}) = \sum_j \alpha_{kj} c(\psi, \omega_{j1}),$$

$$\forall\Omega_1, \Omega_2\colon \Omega_1\,|u_j\rangle = \omega_{j1}\,|u_j\rangle, \Omega_2\,|v_k\rangle = \omega_{k2}\,|v_k\rangle, \forall j \in J, \forall k \in K,$$

(J, K: index sets for the eigenvalues of, respectively, Ω_1, Ω_2; $\alpha_{kj} = \langle v_k | u_j \rangle$: the transformation coefficients).

We shall now show that the construct of a probability tree permits one to perceive in a synthetic way, with intuitive evidence, important features of the quantum mechanical description, some of which remained hidden so far.

The "Potential-Actualization-Actualized" Character of a Probability Tree (Quantum Mechanical Probabilistic Metadependence)

The quantum mechanical transformation theory ($c(\psi, \omega_{k2}) = \sum_j \alpha_{kj} c(\psi, \omega_{j1})$, $\forall \Omega_1, \Omega_2 : \Omega_1 |u_j\rangle = \omega_{j1} |u_j\rangle$, $\Omega_2 |v_k\rangle = \omega_{k2} |v_k\rangle$, $\forall j \in J$, $\forall k \in K$, J, K index sets, $\alpha_{kj} = \langle v_k | u_j \rangle$ the transformation coefficients) permits one to entirely determine from the knowledge of the probability measure $\pi(\psi, \omega_{j1})$ from one branch of a probability tree, any other probability measure $\pi(\psi, \omega_{k2})$ belonging to that same tree. Indeed the equalities $|c(\psi, \omega_{k2}|^2 = |\sum_j \alpha_{kj} c(\psi, \omega_{j1})|^2$, $\forall j \in J$, $\forall k \in K$, are equivalent to the specification of a functional relation (correlation)

$$\pi(\psi, \omega_{k2}) = \mathrm{F_{QM}}[\pi(\psi, \omega_{j1})]$$

between the probability measures $\pi(\psi, \Omega_1)$ and $\pi(\psi, \Omega_2)$ corresponding to the two noncommuting observables Ω_1 and Ω_2. But the standard concept of functional relation between two probability measures does not singularize this particular sort of probabilistic connection between two probability measures introduced by the quantum theory. *Nor does it permit one to recover it fully.*[1] As the index QM emphasizes, we are in the presence of a specifically quantum mechanical functional relation.

This relation can be regarded as a "probabilistic metadependence," in the following sense[1] (pp. 1401–1405): According to the present theory of probabilities the concept of "probabilistic dependence" is by definition confined inside *one* probability space where it concerns *isolated* pairs of *events*. Two events are tied by a "probabilistic dependence" if knowledge of one of these events "influences" the expectations concerning the other one. So the relation $\pi(\psi, \omega_{k2}) = \mathrm{F_{QM}}[\pi(\psi, \omega_{j1})]$ of mutual determination of the probability measures from a quantum mechanical probability tree can naturally be regarded as a "maximal probabilistic metadependence" ("maximal" because it consists of mutual determination; "probabilistic" because, though this determination is a certainty about "influence," nevertheless it concerns probabilistic constructs; "metadependence" because it concerns, not pairs of events from one space, but *globally* pairs of probability measures on entire algebras of events, which, with respect to events, are

metaentities). The notion of a probabilistic metadependence can also be upheld otherwise. Imagine a physicist who does not yet know which state vector $|\psi\rangle$ "describes" the state produced by the operation of state preparation P_ψ. So he makes various measurements on this state in order to establish probability densities that shall determine an adequate mathematical descriptor $|\psi\rangle^{(1)}$ (pp. 1408–1412). Suppose that he decides to work with two noncommuting observables Ω_1 and Ω_2, and, on the basis of some reasons, he envisages two sets of possible probability measures on the corresponding spectra, namely $\sum_1 = \{\pi(\psi, \omega_{m1})\}$ and $\sum_2 = \{\pi(\psi, \omega_{j2})\}$ respectively. (For simplicity suppose that these two sets are discrete). The physicist now asks himself: "What is the (meta)probability for finding, by measurements, this or that probability measure from $\sum_1$ and this or that probability measure from $\sum_2$?" In the absence of any criteria for answering otherwise, he will have to presuppose equipartition on both $\sum_1$ and $\sum_2$. Suppose that he furthermore asks himself: "If for the spectrum $\{\omega_{m1}\}$ of Ω_1 the probability measure were $\pi(\psi, \omega_{m1}) \in \sum_1$, ($k$: known) what would be the corresponding *conditional* probability to find this or that measure $\pi(\psi, \omega_{j2})$ from $\sum_2$ on the spectrum $\{\omega_{j2}\}$ of eigenvalues of Ω_2?". This new question concerns now the product-probability-space where the elementary events are the possible associations $\pi_k(\psi, \omega_{m1})\ \pi(\psi, \omega_{j2})$ between the one measure $\pi_k(\psi, \omega_{m1})$ (supposed known) and the various unknown probability measures envisaged in the set $\sum_2 = \{\pi(\psi, \omega_{j2})\}$. Now: If the physicist ignores the quantum mechanical transformation theory, he must again presuppose equipartition, *which amounts to presupposing independence*, that is, that the probability $\mathrm{P}(\pi_k(\psi, \omega_{m1})\, \pi(\psi, \omega_{j2}))$ of a product-event $\pi_k(\psi, \omega_{m1})\, \pi(\psi, \omega_{j2})$ is the product of the (uniformly distributed) probabilities of the events $\pi_k(\psi, \omega_{m1})$ (fixed) and $\pi(\psi, \omega_{j2})$ (variable inside $\sum_2$). But to this question—which obviously is a *meta* probabilistic question—the quantum mechanical theory of transformations yields another answer. Namely it asserts that the probability measure on the universe of elementary (meta)events $\pi_k(\psi, \omega_{m1})\, \pi(\psi, \omega_{j2})$ is a Dirac dispersion-free measure that associates the probability 1 to that product $\pi_k(\psi, \omega_{m1})\, \pi(\psi, \omega_{j2})$, where $\pi(\psi, \omega_{j2})$ is related with the known measure $\pi_k(\psi, \omega_{m1})$ according to the set of equations $\pi(\psi, \omega_{j2}) = |\sum_m \alpha_{jm} c(\psi, \omega_{m1})_k|^2$, $\forall j \in J$, $\forall m \in M$, (J, M: index sets), and the probability 0 to any other product. This means "dependence," thus showing in what sense the transformation theory can be regarded to assert "probabilistic metadependences".

Now, these probabilistic metadependences between the various probability measures from different branches of a given probability tree *reflect the oneness of the studied state with state vector $|\psi\rangle$ from the common trunk of the tree.* This state that stems from a preparation operation P_ψ and so "is" there, nevertheless "exists" merely as a monolith of still nondifferentiated observational *potentialities*—a monolith of potentialities that sets a genetic unity beneath the various incompatible measurement processes of *actualization* of this or that particular set of observational potentialities, leading to this or that *actualized* observable probability space $[\,]_n$. "$|\psi\rangle$" denotes a class of possible distinct futures of each one individual replica S_i of the studied type of system S.

> *The probability tree of a state with state vector $|\psi\rangle$ is a complex unity which—with respect to the observable manifestations of the studied microsystem—possesses a "potential-actualization-actualized character" ("potential": for $|\psi\rangle$; "actualization," for the measurement evolutions M_Ω; actualized, for the eigenvalues $\omega_j = f_\Omega(V_{j12})$.*

The quantum mechanical functional relations F_{QM} between the probability measures—considered as wholes—from irreducibly distinct observable probability spaces belonging to the same probability tree, reflect the genetic unity of these spaces via the common observational potentialities captured inside the state from the trunk of the tree. The quantum mechanical transformation theory involves new probabilistic features that are neither probabilistic "anomalies" nor mere numerical algorithms. They are a mathematical description of particular realizations of probabilistic meta-properties, brought forth by a growth of the probabilistic thinking that happened inside the process of conceptualization of the microphenomena but have not yet been explicitly exploited in the abstract theory of probabilities.

The Principle of Superposition: A Calculus with Whole Trees

As soon as the principle of superposition comes into play the embeddability into one tree hits a limit. The corresponding quantum mechanical algorithms cease to be embeddable into one single probability tree: *Several trees have to be combined.* The quantum mechanical formalism contains an implicit calculus with *whole probability trees.*

The principle of superposition generates writings of the type $|\psi_{12}\rangle = \lambda_1 |\psi_1\rangle + \lambda_2 |\psi_2|\rangle$ that combine (at least) three trees, namely those introduced by the three operations of state preparation $P_{\psi 12}, P_{\psi 1}, P_{\psi 2}$ corresponding to the three involved state vectors $|\psi_{12}\rangle$, $|\psi_1\rangle$, and $|\psi_2|\rangle$. The *state vectors* $|\psi_{12}\rangle$, $|\psi_1\rangle$, and $|\psi_2|\rangle$ are only *indirectly* concerned by the principle of superposition. This principle concerns directly, essentially, the *operations of state preparation* $P_{\psi 12}$, $P_{\psi 1}$, $P_{\psi 2}$ that produce the states $|\psi_{12}\rangle$, $|\psi_1\rangle$, and $|\psi_2|\rangle$[(1)] (pp. 1405–1424). It asserts that if the two operations $P_{\psi 1}$, $P_{\psi 2}$ are realizable separately, then any operation $P_{\psi 12}$ that is some functional of these operations, $P_{\psi 12} = G(\lambda_1, \lambda_2, P_{\psi 1}; P_{\psi 2})$, *such* that it produces the state $|\psi_{12}\rangle = \lambda_1 |\psi_1\rangle + \lambda_2 |\psi_2|\rangle$ is also realizable. And the probability law for the state $|\psi_{12}\rangle$, for any observable Ω,

$$\pi(\psi_{12}, \omega_j) = |\langle u_j | \psi_{12}\rangle|^2 = |\langle u_j | \lambda_1\psi_1 + \lambda_2\psi_2\rangle|^2$$

compares, *refers* the probability measures from the probability spaces of the *unique* tree obtained when the operation of state preparation $P_{\psi 12} = G(\lambda_1, \lambda_2, P_{\psi 1}, P_{\psi 2})$ is realized, to the corresponding probability measures from the trees that *would* be obtained *if* the operations of state preparation $P_{\psi 1}$ and $P_{\psi 2}$ *were* realized separately. This expresses a peculiar sort of "probabilistic *meta*-metadependence"[(1)] (pp. 1421–1424 and the figures 2, 3A, 3B).

This brings strikingly into evidence that the superposition writings refer to facts that are *fundamentally different* from those concerned by the spectral decomposition writings: The superposition writings concern relations between distinct trees; they qualify in "referred" terms an operation of state preparation and all the probability measures determined by it, for all the dynamical observables. The spectral decomposition writings concern, each, the measurement of only one dynamical observable, on an already prepared state, and they involve only one tree, in a way "nonreferred" to other trees.

Global View on the Probabilistic Organization of Quantum Mechanics

Any observable quantum mechanical elementary event V_j is brought forth by some elementary quantum mechanical chain experiment, some fiber $P_\psi - M_\Omega - V_j$. These fibers are the semantic matter described by the quantum theory. Any fiber $P_\psi - M_\Omega - V_j$ belongs to a probability chain

$$(P_\psi, M_\Omega, \{V_j\}) \rightsquigarrow [\{V_j\}, \tau_F, \pi(P_\psi, M_\Omega, V_j)]$$

In its turn any probability chain belongs to a probability tree $\mathscr{T}(\psi)$, the tree tied with the operation P_ψ of state preparation which starts that chain. So the probability trees define a *partition* on the set of all the chains (hence on the set of all the fibers, hence on the set of all the observable quantum mechanical elementary events V_j).

When one contemplates the landscape determined by this partition each tree appears endowed with its own "internal" calculus (mean value of any dynamical observable Ω with respect to the state vector $|\psi\rangle$ tied with the considered tree, the uncertainty theorem for this state, the principle of spectral decomposition and the predictional probability laws for this state, and the whole quantum mechanical "transformation theory" that relates the probability measures from the different branches of the tree), while the different trees are related by a calculus with whole trees determined by the principle of superposition and the probability law for superposition states.

This is a *hierarchical* view (fibers, chains, trees, connections between trees). It draws attention upon the role played by the *space-time* characteristics of the operations by which the observer produces the objects to be studied (state preparations) and the processes of qualification of these (measurement operations).

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