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A formal system of "questions" and "propositions" conceived by C. Piron and claimed to yield by interpretation quantum mechanics as well as all other known physical theories is examined. It is proved that the mentioned system is syntactically self-consistent in the sense of the theory of models. However, it is found that the mentioned formal system possesses certain syntactic characteristics in consequence of which qualification of this system as a generator of quantum mechanics by interpretation encounters semantic obstacles so grave that they annihilate any relevance of such a qualification.

1. INTRODUCTION

Various authors have tried to produce a formal system of propositions able to yield quantum mechanics by interpretation.(1-4) Among these attempts, the one by Jauch and Piron is certainly the most elaborate and suggestive. However, it has encountered a serious criticism: one of the axioms of the system, asserting the existence of a "product proposition" $a \wedge b$ for any pair (a, b) of propositions from the system, is devoid of semantic definability. Recently Piron has proposed a new formal system of "questions" and "propositions" claimed to eliminate this deficiency. Moreover, this system is claimed to be able to yield by interpretation quantum mechanics as well as any other known physical theory, thus offering a general syntactic scheme for physical theories of any kind.

In this work it will first be shown that Piron's formal system is self-consistent in the sense of the theory of models. However, it will also be shown that its relevance as a syntactic scheme for *physical* theories cannot be accepted. Indeed, by a succession of three theorems it will be brought into evidence that one of the axioms of the system asserts the "existence"

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of a class of propositions for which neither a syntactic method of construction is explicitly available inside the system, nor can a semantic definition be found in consistency with the semantic content assigned to the corresponding descriptive elements from the quantum mechanical formalism. Under such conditions it will be concluded that, as a syntactic scheme for the generation of quantum mechanics by interpretation, the formal system proposed by Piron so far has not attained its aim.

2. THE FORMALISM

We begin by reproducing the formalism to be examined. In order to facilitate any comparison, we give here nearly a literal transcription from Ref. 5 (pp. 19–29). Any reference to another work will be explicitly mentioned. For maximal clarity, we take the liberty of assigning a notation to each of the concepts introduced, without regard to logical dependence.

Definitions, Axioms, Rules, Theorems

D_1 (*physical system*). A physical system is a part of the real world, thought of as existing in spacetime and external to the physicist.

D_2 (*question*). A question is any experiment leading to an alternative of which the terms are "yes" and "no."

D_3 (*opposite or inverse question*). If α is a question, $\alpha^{\sim}$ is the question obtained by exchanging the terms of the alternative.

D_4 (*product of questions*). If $\{\alpha_i\}_{i\in J}$ is a family of questions, $\prod_J \alpha_i$ is the question defined in the following manner: one measures an arbitrary one of the α_i and attributes to $\prod_J \alpha_i$ the answer thus obtained.

Rule R_1 (*opposite of a product question*). By starting from the definitions, one can verify the following rule: $(\prod_J \alpha_i)^{\sim} = \prod_J \alpha_i^{\sim}$.

D_5 (*trivial question*). There exists a trivial question I which consists in measuring anything (or doing nothing) and stating that the answer is "yes" each time.

D_6 (*certain or true question*). When the physical system has been prepared in such a way that the physicist can affirm that in the event of an experiment corresponding to a question α the result will be "yes," the question α is certain or the question α is true.

D_7 (*preorder relation between β and γ*). If the question γ is true whenever the question β is true, the question β is stronger than the question γ, which is symbolized by $\beta < \gamma$. The relation D_7 is transitive.

D_8 (*equivalent questions*). If one has $\beta < \gamma$ and $\gamma < \beta$, then β and γ are equivalent questions, which is denoted by $\beta \approx \gamma$.

D_9 (*proposition*). The equivalence class containing the question β is called *proposition* and is noted by b. The set of all the propositions defined for a system is symbolized by $\mathscr{L}$.

D_{10} (*true proposition*) (see Ref. 6, p. 291). The proposition b is true iff the question β of which b is the equivalence class is true (in the sense of D_6).

Theorem T_1. The set of propositions $\mathscr{L}$ is a complete lattice, i.e., there exists for any family of propositions $\{b_i\}_{i\in J}$ a proposition $\bigwedge_J b_i$ such that

$$x < b_i, \qquad \forall i \in J \Leftrightarrow x < \bigwedge_J b_i$$

We do not reproduce the proof of T_1. However, we remark that the formulation in Ref. 5 of T_1 as well as its proof imply certain (usual) notations and concepts that have not been explicitly defined previously ($\bigwedge_J b_i$, $x < b_i$, $\bigvee_J b_i$ for propositions). The structure of the proof of T_1 implies the assumption of the following well-known meanings for these concepts and notations:

D_{11} (*order relation between propositions*) (Ref. 6, p. 291). If one has $\forall\beta \in b, \forall\gamma \in c\colon \beta < \gamma$, then the proposition b is stronger than the proposition c, which is symbolized by $b < c$.

D_{12} (*"product" or "conjunction" of propositions*). Given any family of propositions $\{b_i\}_{i\in J}$ from $\mathscr{L}$, $\bigwedge_J b_i$ denotes the equivalence class containing the question $\prod_J \beta_i$, where $\beta_i \in b_i$.

D_{13} (*"sum" of propositions*). Given a family $\{b_i\}_{i\in J}$ of propositions from $\mathscr{L}$, $\bigvee_J b_i$ denotes the product $\bigwedge_\alpha x_\alpha$ of all the propositions $x_\alpha \in \mathscr{L}$ such that $b_i < x_\alpha$, $\forall i$.

D_{14} (*absurd proposition, trivial proposition*).[2] Theorem T_1 entails the existence of an absurd proposition $\bigwedge_{b\in\mathscr{L}} b = 0$. The equivalence class of the trivial question I defines a trivial proposition I (same notation as for the trivial question).

D_{15} (*complementary proposition for b*). The proposition c is a complementary proposition for a given proposition b if $b \vee c = 1$ and $b \wedge c = 0$.

D_{16} (*compatible complement for b*). The proposition c is a compatible complement $c = b'$ of a given proposition b if it is a complementary proposition for b and if furthermore there exists a question β such that $\beta \in b$ and $\beta^{\sim} \in c$.

[2] Not explicitly defined in Ref. 5.

Axiom C (*existence of a compatible complement*). For each proposition b there exists at least one compatible complement b'.

The well-known concepts of a lattice and of lattice generated by a family of propositions are then used for the following axiom:

Axiom P. If $b < c$ are propositions from $\mathscr{L}$ and if b' is the compatible complement for b, and c' is the compatible complement for c, then the sublattice generated by (b, b', c, c') is a distributive lattice.

Axiom P entails: (1) the uniqueness of the compatible complement b' for any $b \in \mathscr{L}$; (2) orthocomplementation:

$$\forall b \in \mathscr{L}: \quad (b')' = b, \quad b \vee b' = I, \quad b \wedge b' = 0$$

$$\forall (b, c) \in \mathscr{L} \times \mathscr{L}: \quad b < c \Rightarrow b' > c'$$

and (3) weak modularity:

$$\forall (b, c) \in \mathscr{L} \times \mathscr{L}, \quad \text{if } b < c, \quad \text{then } c \wedge (c' \vee b) = b, \quad b \vee (b' \wedge c) = c$$

D_{17} (*atoms*). If p is such that $0 < x < p \Rightarrow x = 0$ or $x = p$, then p is called an atom.

Axiom A (*atomicity, covering law*): If $b \in \mathscr{L}$, $b \neq 0$, then there exists an atom $p: p < b$. If p is an atom and if $p \wedge b = 0$, then $p \vee b$ covers b (in the sense of a well-known definition, Ref. 7, p. 848).

Finally, two further definitions are relevant for the subsequent study:

D_{17} (*orthogonal propositions*): $b \in \mathscr{L}$ is orthogonal to $c \in \mathscr{L}$ and is symbolized by $b \perp c$ if $b < c'$.

D_{18} (*propositional system*). A complete lattice satisfying axioms C, P, and A is a propositional system.

3. THE STUDY

The formalism briefly reproduced above is built on two interconnected levels, the level of questions and the level of propositions. For this reason we call it a questions–propositions system and symbolize it by the notation qp-s. Now, on the level of *propositions*,[3] the logicomathematical structure which emerges is that of a complete, orthocomplemented, weakly modular, and atomic lattice (D_{18}). Such a logicomathematical structure is known to be isomorphic to one introduced by the Hilbert space formulation of quantum mechanics and consequently it is known to be a formally self-consistent structure.[2,3] But the global structure introduced by the *two* levels, of propositions *and* of questions, interconnected according to the definitions D_9–D_{16} is a new structure, which involves more basic assertions than the usual lattice-theoretic formulations, namely those concerning relations between questions and propositions. As far as we know, such a formal structure has not yet been studied. In particular, it is not at all obvious a priori that this structure is formally self-consistent. Indeed, the definition D_4 of a product of questions and the rule for the "negation" of such a product $(\prod_J \alpha_i)^\sim = \prod_J(\alpha^\sim)$ are unusual and seem to contradict de Morgan's law.

[3] In the sense of $(D_9 + D_2)$.

In what follows we shall first show that the qp-s is self-consistent in the sense of the abstract theory of models, i.e., we shall show that it does admit a model.[8]

However, we shall furthermore show that the Axiom C associated with the definition D_4 for a product of questions, with the definition D_3 of the opposite question and with the consequent rule R_1 for the opposite of a product of questions, raises a very grave formal problem, which is the source of a semantic barrier in the way of the interpretability of the qp-s as a physical theory.

3.1. Existence of a Model for the qp-s Formalism

According to the theory of models a formal system is proved to be self-consistent if a model is produced for it, i.e., if a "realization" of the language of this formal system is produced which validates all the axioms of the system. By definition, a realization of a formal system is a set $\mathscr{S}$ and certain relations and operations in $\mathscr{S}$ yielding an interpretation of the functional symbols and of the predicates from the studied formal system.

In the present case one has to define a set Q with a preorder relation and the following two operations:

(1) $\alpha \in Q \rightarrow \alpha^\sim \in Q$.

(2) $\{\alpha_i\}_J \in P(Q) \rightarrow \prod_J \alpha_i \in Q$.

$P(Q)$ being the power set of Q, and such that:

(A) $(\alpha^\sim)^\sim = \alpha$.

(B) $(\prod_J \alpha_i)^\sim = \prod_J(\alpha^\sim)$.

(C) The preorder relation defines, by passage to equivalence classes, a complete, atomic lattice $\mathscr{L}$ satisfying the covering law.

(D) $\mathscr{L}$ is provided with an orthocomplementation $a \rightarrow a'$ such that

$$\forall a \in \mathscr{L}, \exists \alpha \in a: \quad \alpha^\sim \in a'$$

This makes $\mathscr{L}$ a weakly modular lattice.

(E) For any family $\{a_i\}_{i \in J}$ of elements of $\mathscr{L}$ and for any choice $\alpha_i \in a_i$, we have

$$\prod_J \alpha_i \in \bigwedge_J a_i$$

Now, we assert the following:

Theorem $\mathscr{T}_1$. The qp-s does admit a model.

Proof. Given in Appendix, because of its length.

The conclusion imposed by Theorem $\mathscr{T}_1$ is far from being trivial, for two distinct reasons which might be related. In the first place, as soon as one realizes fully the unusual character of the definition for a product question, the existence of at least a certain sort of self-consistency for the qp-s appears as surprising much more than as natural. In the second place, the model constructed in the proof possesses certain striking peculiarities: While the propositions are represented by closed subspaces of a Hilbert space, a product of questions is represented by a sum (of sets of subspaces). This suggests that the formal self-consistency proved with the help of a such a drastic distortion might somehow lead to difficulties in also mimicking the *semantic* structure associated to the quantum mechanical formalism.

3.2. Critique

As far as we know, Axiom C has not yet been seriously criticized. We believe that this acceptance of Axiom C stems from a false assumption. Namely, several authors (for example, Greechie and Gudder,[9] Mišra,[10] and, in an early version of qp-s, Jauch and Piron[11]) consider that the compatible complement a' of a proposition of a qp-s simply consists of the class $\{\alpha^\sim\}$ of all the negations $\alpha^\sim$ of the questions $\alpha \in a$.[4] If this were true, Axiom C would be so trivially satisfied that the necessity of its statement as an axiom would be questionable. But in fact the mentioned assumption is not true, and what *is* questionable is the very "existence" of a compatible complement a' for any $a \in \mathscr{L}$. This will be shown now by a succession of three increasingly far-reaching theorems.

Small Theorem $\mathscr{S}\mathscr{T}$.[5] Inside the qp-s there exists at least one proposition such that its compatible complement is different from the class of all the negations of all the questions of which that proposition is the equivalence class.

[4] There exists in fact a critical attempt by Mielnik[12] but which is viciated by the same false assumption.

[5] We find no better name for this first assertion, which is much less important than the two subsequent ones; it is neither a lemma nor a consequence; as to the word "proposition," it would be confusing in the present context.

Proof. By construction: Given any $a \in \mathscr{L}$, let us form the proposition $a \wedge a' = 0$. It is possible to define constructively the compatible complement $0'$ of 0: According to the requirement D_{15}, the unique complement for 0 is I. So I also has to be the compatible complement required for 0 by Axiom C. Now,

$$\exists \alpha \in a: \quad \alpha^\sim \in a' \Rightarrow \alpha \,\pi\, a^\sim \in a \wedge a' = 0$$

and

$$(\alpha \,\pi\, \alpha^\sim)^\sim = \alpha^\sim \,\pi\, \alpha \in a' \wedge a = 0 \neq I = 0' \quad \text{QED}$$

This situation can be illustrated by the following "noncommutative" diagram:

$$\exists a \in \mathscr{L}, \qquad \exists \alpha \in a: \quad \begin{array}{ccc} \alpha & \in & a \\ \updownarrow & & \updownarrow \\ \alpha^\sim & \notin & a' \end{array}$$

The above example introduces the highly particular proposition 0 (moreover, for this proposition it has been possible to construct a compatible complement). But we shall now prove a theorem concerning *any* proposition from qp-s:

Theorem $\mathscr{T}_2$. For any $a \in \mathscr{L}$ distinct from the trivial proposition, the compatible complement a' is different from the class of the negations of all the questions of which a is the equivalence class.

Proof. Given any $a \in \mathscr{L}$, $a \neq I$, we choose any question b orthogonal to a but distinct from a'. We shall show by construction that there exists a question $\alpha \in a$ such that $\alpha^\sim \in b$. Since each question belongs to one and only one proposition, this will prove that $\alpha^\sim \notin a'$, which will establish $\mathscr{T}_2$.

By definition D_{16} and Axiom C there exist inside the qp-s questions (α, β) such that $\alpha \in a$, $\alpha^\sim \in a'$, $\beta \in b$, $\beta^\sim \in b'$.

Consider now the question $\gamma = \alpha \,\pi\, \beta^\sim$. By definition D_{17} we have $a < b'$, hence $b < a'$. One then has $\gamma = \alpha \,\pi\, \beta^\sim \in a \wedge b' = a$, whereas $\gamma^\sim \in (\alpha \,\pi\, \beta^\sim)^\sim = \alpha^\sim \,\pi\, \beta \in a' \wedge b = b \neq a'$.

Thus $\gamma \in a$ and $\gamma^\sim \in b \neq a'$. This proves $\mathscr{T}_2$.

The content of theorem $\mathscr{T}_2$ can be graphically expressed as follows:

$$\forall a \in \mathscr{L}, \qquad \exists \alpha: \quad \begin{array}{ccc} \alpha & \in & a \\ \updownarrow & & \updownarrow \\ \alpha^\sim & \notin & a' \end{array}$$

Interpretative Illustration. Consider a particle in a one-dimensional space, say represented by the real line. Let α, β be two questions defined as follows: α is defined by an apparatus capable of verifying whether or not the position of the particle belongs to the set (0, 1), the answer "yes" corresponding to the case that the particle is found in (0, 1); analogously, β is an apparatus capable of verifying if the position of the particle belongs to the set (2, 3). Note that, by definition D_3, $\alpha^\sim$ and $\beta^\sim$ correspond respectively to the same apparatus, but for these questions the answers "yes" and "no" have been inverted. Let a, b be the propositions to which α, β belong. We note that in this illustration, apart from their syntactic definitions, the propositions a, b are furthermore endowed with a semantic content: a, for instance, is here the proposition "the position of the particle belongs to the space interval represented by (0, 1)," which can be verified or falsified by use of the apparatus α. By construction we have $\alpha \in a$, $\alpha^\sim \in a'$, $\beta \in b$, $\beta^\sim \in b'$. Furthermore, $b < a'$. Thus, by definition D_{17}, b and α are orthogonal.

If we now define the question $\gamma = \alpha \,\pi\, \beta^\sim$, then, as shown in the proof of Theorem $\mathscr{T}_2$, one has $\gamma \in \alpha$, $\gamma^\sim \notin a'$. Thus—syntactically—we are already in the presence of the surprising fact that the negation of a given question lies somewhere *outside* the complement of the proposition to which that question belongs. But let us now get down to semantics. Suppose that we measure $\gamma \in \alpha$ and that we find the answer "yes": In general, no conclusion whatever can be drawn therefrom concerning the truth of the proposition a, namely "the position belongs to (0, 1)," despite the fact that $\gamma \in a$: Indeed, by definition D_4, in order to perform a measurement of γ, we have to choose one of the two questions $\alpha, \beta^\sim$. If by chance we choose to measure $\beta^\sim$, then an answer "yes" for γ means, by definitions D_3 and D_4, that the answer to β is "no," that is, that the particle has been found in $(\mathbb{R} - (2, 3))$. However, *exclusively* from *this* information, obviously it cannot be decided whether a is true or not. This brings into evidence the fact that the formal characteristic of the qp-s expressed by Theorem $\mathscr{T}_2$ in general entails certain specific awkward semantic consequences in an interpretation of qp-s.

Thus, contrary to a widely held opinion, the compatible complement a'—quite generally—cannot be formed as the class of all the negations of the questions from a qp-s. This, however, does not yet lead to doubts about the existence of a'. Indeed, so far there is still the possibility that for each given $a \in \mathscr{L}$ some method for constructing a nonvoid a' is specifiable, even if a' does not contain *all* the negations α' of the $\alpha \in a$. But we shall now prove a theorem which seems to deny this possibility also.

Let us consider any pair of distinct propositions $a \in \mathscr{L}$, $b \in \mathscr{L}$, $a \neq b$, $a = \{\alpha_i\}_{i \in I}$, $b = \{\beta_j\}_{j \in J}$. Furthermore, let us consider all the product questions $\alpha \,\pi\, \beta$, $\alpha \in a$, $\beta \in b$. By definition D_{12} the equivalence class of $\alpha \,\pi\, \beta$ defines the proposition $a \wedge b$. Now we assert that the compatible complement $(a \wedge b)'$ of $a \wedge b$ contains *none* of the negations $(\alpha \,\pi\, \beta)^\sim$ of the questions $(\alpha \,\pi\, \beta)$ by help of which—exclusively—the proposition $a \wedge b$ can be *defined*:

Theorem $\mathscr{T}_3$. Given in $\mathscr{L}$ the proposition $a \wedge b$, $(a, b) \in \mathscr{L}^2$, $a \neq b$, defined as the equivalence class of any question $\alpha \,\pi\, \beta$, $\alpha \in a$, $\beta \in b$, one has $(\alpha \,\pi\, \beta)^\sim \notin (a \wedge b)'$, $\forall (\alpha \,\pi\, \beta)$.

Proof. By reduction to absurdity: Let us suppose the contrary, namely that there exists at least one pair of questions α and β, $\alpha \in a$ and $\beta \in b$, such that $(\alpha \,\pi\, \beta)^\sim \in (a \wedge b)'$. Let c and d be the propositions containing $\alpha^\sim$ and $\beta^\sim$, respectively: $\alpha^\sim \in c$, $d^\sim \in d$. Then we find

$$(\alpha \,\pi\, \beta)^\sim = \alpha^\sim \,\pi\, \beta^\sim \in c \wedge d$$

Using the notation $\vee$ for the disjunction in the lattice $\mathscr{L}$, we can write

$$(a \wedge b)' = a' \vee b'$$

so that

$$c \wedge d = a' \vee b'$$

Thus

$$c > a', \qquad c > b', \qquad d > a', \qquad d > b' \tag{1}$$

Now, Axiom P together with the inequality $c > a'$ entails that the lattice generated by (a, a', c, c') is distributive. Consequently,

$$(a \wedge c) \vee a' = (a \vee a') \wedge (c \vee a') = I \wedge c = c \tag{2}$$

Because $\alpha \,\pi\, \alpha^\sim$ is never "true," we have

$$\alpha \,\pi\, \alpha^\sim \in 0$$

but also

$$\alpha \,\pi\, \alpha^\sim \in a \wedge c$$

so that

$$a \wedge c = 0$$

Thus

$$(a \wedge c) \vee a' = 0 \vee a' = a' \tag{3}$$

and relations (2) and (3) entail

$$c = a' \tag{4}$$

By a similar inference, we can find

$$d = b' \tag{5}$$

Relations (1), (4), and (5) entail

$$a' = b'$$

so that

$$a = b$$

contrary to the initial assumption that a and b are nonidentical propositions.

Thus the hypothesis that there exists at least one pair of questions $\alpha \in a$, $\beta \in b$ such that $(\alpha \pi \beta)^{\sim} \in (a \wedge b)'$ is false. This proves Theorem $\mathcal{T}_3$.

The content of Theorem $\mathcal{T}_3$ can be expressed graphically as follows:

$$\begin{array}{ll} \forall (a, b) \in \mathcal{L} \times \mathcal{L}, a \neq b & \alpha \pi \beta \;\in\; a \wedge b \\ & \;\updownarrow \qquad\quad \updownarrow \\ \forall \alpha \pi \beta \in a \wedge b & (\alpha \pi \beta)^{\sim} \notin (a \wedge b)' \end{array}$$

Illustrative Challenge. Consider a microsystem S of *non*zero mass. Consider also two apparatus $\mathcal{A}(Q)$ and $\mathcal{A}(P)$ utilizable for measuring—in the sense of the quantum mechanical theory of measurement—respectively the position observable Q of S and the momentum observable P of S, which, as is known, are noncommuting: $[Q, P] \neq 0$. Now, inside the qp-s, for *any* system and *any* apparatus able to measure something on this system, there exist questions corresponding to these possible measurements on this system. So in particular for the microsystem S and the apparatus $\mathcal{A}(Q)$ there exist inside qp-s certain questions α, while for S and the apparatus $\mathcal{A}(P)$ there exist inside qp-s certain questions β. Furthermore, inside the qp-s, each question q defines (as its equivalence class) a certain proposition p. Let then $a \in \mathcal{L}$ and $b \in \mathcal{L}$ be, respectively, two propositions defined by two chosen questions α and β. Each pair of propositions from $\mathcal{L}$ defines a corresponding product proposition belonging to $\mathcal{L}$, so in particular one can write $a \wedge b \in \mathcal{L}$. Finally, Axiom C asserts the existence inside $\mathcal{L}$ of the compatible complement p' of any proposition $p \in \mathcal{L}$, hence also, in particular, according to Axiom-C, $(a \wedge b)' \in \mathcal{L}$ does exist. But according to Theorem $\mathcal{T}_3$ *none* of the negations $(\alpha_i \pi \beta_j)'$ of the questions $\alpha_i \pi \beta_j$, $\alpha_i \in a$, $\beta_j \in b$ does belong to $(a \wedge b)'$. Our challenge then is the following one:

1. Try to specify *syntactically* inside the qp-s at least one question belonging to $(a \wedge b)'$, i.e., try to produce by the qp-s definitions, rules, theorems, and axioms an algorithm constructing a question of some structure—necessarily different from $(\alpha \pi \beta)^{\sim}$, $\alpha \in a$, $\beta \in b$—which certainly does belong to $(a \wedge b)'$ by syntactic necessity. (Note that neither $a \wedge b$ nor $(a \wedge b)'$ possesses a defined correspondent inside quantum mechanics.)

2. Specify the apparatus corresponding to this question [note that such an apparatus would have to be different from both $\mathcal{A}(Q)$ and $\mathcal{A}(P)$, while inside quantum mechanics *only* these apparatus are considered in connection with the "position" and the "momentum" of S].

In case of success it will have been shown that, notwithstanding Theorem $\mathcal{T}_3$, Axiom C is endowed inside the qp-s with some specifiable "syntactic meaning," *even* if this syntactic meaning does *not* generate a

semantic meaning defined inside quantum mechanics when quantum mechanics is researched as an interpretation of the qp-s.

But in case of failure we are once more in the presence of a formal system subject to precisely the *same* sort of criticism which has already been given of the former Jauch–Piron formal system: Again the propositions of the type $a \wedge b$ are the source of problem, this time via Axiom C, which asserts the "existence" of a compatible complement for *any* proposition from $\mathcal{L}$, hence also when, in particular, this proposition is of the form $a \wedge b$, a and b being equivalence classes of questions consisting respectively of two measurements of two noncommuting quantum mechanical observables. Indeed, what *sort* of a *meaning* could be assigned to such a compatible complement, in a case in which no method of syntactic construction can be specified for at least one question from the class of which this asserted complement consists, and in which no semantic definition can be assigned to this class in coherence with the semantics associated with the quantum mechanical formalism inside the quantum theory, of which the qp-s is claimed to be an adequate generator by interpretation? And if *no sort* of meaning whatever can be assigned to $(a \wedge b)' \in \mathcal{L}$, if it is merely an intermediary formal entity somehow "useful" inside the qp-s, then how could it be accepted that such an entity be introduced by one of the *fundamental axioms* of a formal system built with the *aim* of yielding by interpretation the *physical* theory denominated quantum mechanics?

In this sense Theorem $\mathcal{T}_3$ raises a strong doubt concerning both the "existence" asserted by Axiom C and the relevance of the qp-s as a generator of quantum mechanics by interpretation.

4. CONCLUSION

We have shown that the questions–propositions system qp-s proposed by Piron is self-consistent in the sense of the theory of models. But we have also shown by a sequence of three increasingly wide-reaching theorems that the relevance of this formal system as a generator of quantum mechanics by interpretation is strongly questionable: One of the axioms of this system asserts the "existence" of a type of element belonging to the system for which in certain cases neither a syntactic method of construction is specified inside the system nor—a fortiori—a possible semantic definition is indicated in consistency with the semantic structure associated with the quantum mechanical formalism inside the quantum theory. In such circumstances it seems very difficult indeed to retain the contention that the qp-s is a formal structure which is able to "yield" quantum mechanics by interpretation.

This structure, on the contrary, seems to be *fundamentally* inadequate for *this* aim.

More generally, the group of theorems proved in this work brings strikingly into evidence the fact, probably generally known but certainly often neglected, that a formal system, if it is built with the aim of yielding by interpretation a given mathematical description of a domain of reality, has to satisfy constraints that are in general different from a mere isomorphism with the mathematical structure used in this description. *What* these constraints are, is an important nonelucidated question.

APPENDIX

Theorem $\mathscr{T}_1$. The questions–propositions system has a model.

Proof. Let H be a separable Hilbert space and L be the set of all closed subspaces of H. As is well known (Section 2), L is a complete, atomic, orthocomplemented and weakly modular lattice satisfying the covering law. We shall denote in L the order relation by $\prec$, the orthocomplementation by a prime, and the conjunction by $\curlywedge$. Let Q be the power set of L,

$$Q = \text{def } P(L) \tag{A1}$$

For any $\gamma \in Q$ we define $A(\gamma) \in L$ by the relation

$$A(\gamma) = \text{def} \bigwedge_{x\in\gamma} x \tag{A2}$$

furthermore, we can define the preorder relation on Q by

$$\gamma < \delta \Leftrightarrow \text{def } A(\gamma) \prec A(\delta) \tag{A3}$$

We define the two other applications $\alpha \to \alpha^{\sim}$ and $\{\alpha_i\}_{i\in J} \to \prod_J \alpha_i$ by the formulas

$$\alpha^{\sim} = \text{def}\{x' : x \in \alpha\} \tag{A4}$$

$$\prod_J \alpha_i = \text{def} \bigcup_J \alpha_i \tag{A5}$$

where $\bigcup$ is simply the set-theoretic union. Definitions (A1) and (A3)–(A5) form a realization of the theory. We shall now prove that this realization is a model, i.e., that conditions (A) $\to$ (E) are satisfied (Section 3.1).

(a) Condition (A) is an immediate consequence of (A4).

(b) Condition (B) can be proved easily[6]:

$$\left(\prod_J \alpha_i\right)^{\sim} \overset{(A4)}{=} \left(x' : x \in \prod_J \alpha_i\right) \overset{(A5)}{=} \left(x' : x \in \bigcup_J \alpha_i\right)$$
$$= \bigcup_J \{x' : x \in \alpha_i\} \overset{(A4)}{=} \bigcup_J \alpha_i^{\sim} \overset{(A5)}{=} \prod_J \alpha_i^{\sim}$$

(c) To prove condition (C), we proceed as follows. First, as is easily verified, the relation $<$ is a reflexive and transitive one, thus a preorder relation: consequently the relation defined on Q by

$$\gamma \approx \delta \Leftrightarrow \text{def } \gamma < \delta \quad \text{and} \quad \delta < \gamma \tag{A6}$$

is an equivalence relation. Hence it defines the set $\mathscr{L}$ of equivalent classes of elements of Q and an order relation on $\mathscr{L}$:

$$\forall a, b \in \mathscr{L}: \quad a \subseteq b \Leftrightarrow \text{def } \alpha < \beta, \quad \forall \alpha \in a, \quad \forall \beta \in b \tag{A7}$$

In addition, relations (A2), (A3), and (A6) show that $\gamma \approx \delta$ iff $A(\gamma) = A(\delta)$. In other terms $A(\gamma) = A(\delta)$ iff γ and δ belong to the same class a. Hence we can write

$$\forall a \in \mathscr{L}: \quad A(a) = \text{def } A(\alpha), \quad \alpha \in a \tag{A8}$$

Thus the A defined is an application $\mathscr{L} \to L$. This application is injective:

$$A(a) = A(b) \overset{(A8)}{\Rightarrow} A(\alpha) = A(\beta), \quad \forall \alpha \in a, \quad \beta \in b$$
$$\Rightarrow \alpha \approx \beta \Rightarrow a = b$$

Furthermore, if $x \in L$, then $\{x\} \in Q$ and we have, by relation (A2),

$$A(\{x\}) = x$$

If a is the equivalence class containing $\{x\}$, then $A(a) = A(\{x\}) = x$. Hence

$$\forall x \in L, \quad \exists a \in \mathscr{L}: \quad A(a) = x$$

so that A is also a surjective application. Thus finally relation (A7) implies that

$$A(a) \prec A(b) \Leftrightarrow A(\alpha) \prec A(\beta), \forall \alpha \in a, \beta \in b \Leftrightarrow \alpha < \beta \Leftrightarrow a \subseteq b$$

[6] A number above a symbol indicates the relation in consequence of which this symbol can be written.

i.e., the application A is a bijection, preserving the order relation between the ordered sets L and $\mathscr{L}$ (an isomorphism). Since L is a complete atomic lattice satisfying the covering law, $\mathscr{L}$ also will possess that structure. In addition, we can define an orthocomplement on $\mathscr{L}$ by the relation

$$\forall a \in \mathscr{L}: \quad a' = \text{def } A^{-1}[(A(a))'] \tag{A9}$$

It can be easily verified that A also has properties of orthocomplementation and weak modularity: condition (C) is thus satisfied.

(d) Concerning condition (D), we only have to prove that

$$\forall a \in \mathscr{L}, \quad \exists \alpha \in a: \quad \alpha^{\sim} \in a'$$

First we note that, for any $a \in \mathscr{L}$, one can verify the following relations:

$$A(a') \overset{(A9)}{=} A[A^{-1}(A(a)')] = [A(a)]' \tag{A10}$$

$$A(a) \in L \Rightarrow \{A(a)\} \in Q \overset{(A2)}{\Rightarrow} A(\{A(a)\}) = A(a) \Rightarrow \{A(a)\} \in a \tag{A11}$$

$$\{A(a)\}^{\sim} \overset{(A4)}{=} \{(A(a))'\} \overset{(A10)}{=} \{A(a')\} \tag{A12}$$

Then, since (A11) holds for any element a, one has

$$\{A(a')\} \in a' \tag{A13}$$

and relations (A12) and (A13) imply

$$\{A(a)\}^{\sim} \in a' \tag{A14}$$

Finally, relations (A11) and (A14) show that, for any $a \in \mathscr{L}$ there exists at least one $\alpha \in Q$, namely $\alpha = \{A(a)\}$, such that $\alpha^{\sim} \in a'$: condition (D) is satisfied.

(e) As for condition (E), given any family $\{a_i\}_{i \in J}$ of elements of $\mathscr{L}$ and any $\alpha_i \in a_i$, we find, since A is an isomorphism, that

$$A\left(\bigwedge_J a_i\right) = \bigwedge_J A(a_i) \tag{A15}$$

and

$$A\left(\prod_J \alpha_i\right) \overset{(A5)}{=} A\left(\bigcup_J \alpha_i\right) = \bigwedge_{x \in \bigcup_J \alpha_i} x = \bigwedge_J \bigwedge_{x \in \alpha_i} x = \bigwedge_J A(\alpha_i) = \bigwedge_J A(a_i) \tag{A16}$$

From equalities (A15) and (A16), we infer

$$A\left(\bigwedge_J a_i\right) = A\left(\prod_J \alpha_i\right) \Rightarrow \prod_J \alpha_i \in \bigwedge a_i$$

which suffices to show that condition (E) also is satisfied, and achieves the proof of the theorem.

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